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Work during this period had been concerned with: 1) optimal maneuvering of flexible spacecraft and 2) control of damped distributed structures. The problem of simultaneous maneuvering and vibration control is solved by means of a perturbation approach. The control policy for the perturbations involves the solution of a time-varying linear regulator problem capable of accomodating persistent disturbances. Control of self-adjoint (undamped) distributed structures can be carried out conveniently by modal control. Nonproportional damping tends to destroy the self-adjointness of the system, so that modal control is not as convenient as for undamped structures. If damping is relatively small, however, it is possible to base the control design on the self-adjoint system and still obtain satisfactory control performance.

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Abstract

Work on the grant has concentrated on two aspects: 1) optimal maneuvering of flexible spacecraft and 2) control of damped distributed structures.

The problem of simultaneous maneuver and vibration control of flexible spacecraft is solved by means of a perturbation approach whereby the slewing of the spacecraft regarded as rigid represents the zero-order problem and the control of vibration, as well as perturbations from the rigid-body maneuver, represents the first-order problem. The perturbation approach to maneuvering of flexible spacecraft was developed earlier by the author of this report. However, the present control design is different from the earlier design, particularly the feedback control of the perturbations. The zero-order problem is solved according to a minimum-time policy, resulting in bang-bang control. The control of the elastic vibration and of the perturbations from the rigid-body maneuver is carried out simultaneously with the "rigid-body" slewing, rather than sequentially, so that the overall result is very close to an ideal minimum-time solution. The maneuver control is open-loop. On the other hand, the control policy for the perturbations involves the solution of a time-varying linear regulator problem capable of accommodating the disturbances from the rigid-body maneuver. An exponential factor in the performance index forces the error to zero within the minimum-time maneuver period. Feedback control of the perturbations is essential in view of possible inaccuracies in the system parameters affecting the natural frequencies and mode shapes. The work is described in Ref. 1.

Undamped distributed structures represent self-adjoint systems,

admitting real eigenvalue and real orthogonal eigenfunctions. Control of self-adjoint systems can be carried out conveniently by modal control. Distributed structures with proportional damping possess the same eigenfunctions as the corresponding undamped structures, so that the same modal approach can be used in this case as well. Nonproportional damping tends to destroy the self-adjointness of the system, so that modal control is not as convenient as for undamped structures. If damping is relatively small, however, it is possible to base the control design on the self-adjoint system and still obtain satisfactory control performance (Ref. 2).

References (Enclosed)

1. Meirovitch, L. and Sharony, Y., "Optimal Vibration Control of a Flexible Spacecraft During a Minimum-Time Maneuver," Proceedings of the Sixth VPI&SU/AIAA Symposium on Dynamics and Control of Large Structures, Blacksburg, VA, June-July 1987, pp. 579-601.

2. Meirovitch, L. and Norris, M. A., "Control of Distributed Structures with Small Nonproportional Damping," Proceedings of the AIAA Guidance, Navigation and Control Conference, Monterey, CA, August 1987, pp. 99-105.

Optimal Vibration Control of a Flexible Spacecraft During a Minimum-Time Maneuver

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Abstract

This paper is concerned with the simultaneous maneuver and vibration control of a flexible spacecraft. The problem is solved by means of a perturbation approach whereby the slewing of the spacecraft regarded as rigid represents the zero-order problem and the control of vibration, as well as of perturbations from the rigid-body maneuver, represents the first-order problem. The zero-order control is to be carried out in minimum time, which implies bang-bang control. On the other hand, the first-order control is a time-dependent linear quadratic regulator including integral feedback and prescribed convergence rate.

1. Introduction

The problem of slewing a flexible spacecraft is described by a hybrid set of equations, in the sense that the rigid-body motions of the spacecraft are described by ordinary differential equations, generally nonlinear, and the elastic motion of the flexible parts by partial differential equations. Practical considerations dictate that the partial differential equations be discretized in space, resulting in a set of nonlinear ordinary differential equations of relatively high order.

The problem has been considered by Junkins and Turner (Ref. 1), Turner and Chun (Ref. 2) and Chun, Turner and Juang (Ref. 3). The approach of Refs. 1-3 was to minimize a quadratic performance index for the nonlinear discretized system. The problem was characterized by prescribed end position and fixed final time. A different approach, suggested by Ben-Asher, Burns and Cliff (Ref. 4) and by Thompson, Junkins and Vadali (Ref. 5), consists of letting the final time be free and solving a minimum-time problem for the nonlinear model. Both approaches involve the solution of a nonlinear two-point boundary-value problem for a high-order system. In the second approach, the control is open-loop.

A different approach was developed by Meirovitch and Quinn (Refs. 6 and 7). The approach is based on the perturbation concept for solving

nonlinear differential equations. The zero-order problem consists of the rigid-body maneuver of the flexible spacecraft, and is described by six nonlinear ordinary differential equations. The first-order problem is obtained by linearizing the system of equations about the trajectory describing the maneuver. The perturbation equations represent a high-order set of linear time-varying ordinary differential equations. The rigid-body maneuver is open-loop and the control of the perturbed system is closed-loop. Whereas the feedback control is carried out during the maneuver, the bulk of the vibration suppression takes place after the termination of the maneuver.

This paper adopts the perturbation approach of Refs. 6 and 7, but the control design is different, particularly the feedback control of the perturbations. The zero-order problem is solved according to a minimum-time policy, resulting in bang-bang control. The control of the elastic vibration and of the perturbations from the rigid-body maneuver is carried out simultaneously with the "rigid-body" slewing, so that the overall result is very close to an ideal minimum-time motion. The maneuver control is open-loop. On the other hand, the control policy for the perturbations involves the solution of a time-varing linear regulator problem (LQR) capable of accommodating the disturbances caused by the rigid-body maneuver. An exponential factor in the performance index forces the error to zero within the minimum-time maneuver period. Feedback control for the perturbations is essential in view of possible inaccuracies in the system parameters affecting the natural frequencies and mode shapes.

2. The Equations of Motion

We consider a flexible spacecraft consisting of a rigid hub and a flexible appendage, as shown in Fig. 1. For convenience, we introduce an inertial reference frame XYZ and a set of axes xyz embedded in the spacecraft in undeformed state, so that x coincides with the axis of the undeformed appendage. We shall refer to xyz as body axes. The origin O of the body axes coincides with mass center of the spacecraft in undeformed state. The general motion of the spacecraft can be described in terms of the translation and rotation of the body axes relative to the inertial frame and the elastic motion of the appendage relative to the body axes. To this end, we denote by $\underset{\sim}{R}$ the position vector of O relative to the inertial space, by $\underset{\sim}{r}$ the nominal position vector of a point in the spacecraft relative to the body axes and by $\underset{\sim}{u}(\underset{\sim}{r},t)$ the elastic displacement vector of a typical point in the appendage. Moreover, we denote by $\underset{\sim}{\Omega}(t)$ the angular velocity of the body axes relative to the inertial space. The equations of motion represent a hybrid set of differential equations consisting of six ordinary differential equations for the rigid-body motion of the body axes and three partial differential equations for the elastic motion relative to the body axes. The partial differential equations for $\underset{\sim}{u}$ can be replaced by 3N ordinary differential equations, where N is the number of degrees of freedom used to

represent each component of $\mathbf{u}$. Reference 6 presents the complete set of equations. Our interest is in a special type of motion, so that the general equations are not listed here.

Let us consider the single-axis maneuver of the spacecraft, and in particular the slewing of the spacecraft in the yz-plane, which amounts to the rotation about the x-axis through a given angle. Ideally, the maneuver is to be carried out as if the spacecraft were rigid. In practice, a pure rigid-body maneuver is difficult to achieve, so that the maneuver will excite elastic motions, which in turn will cause the spacecraft to deviate from the rigid-body maneuver. Due to the configuration of the spacecraft under consideration (see Fig. 1), the equations for the rigid-body and elastic displacements in the yz-plane and the rotation about the x-axis are decoupled from the remaining equations of motion. Because these equations correspond to the very same motions involved in the slewing maneuver, we concentrate on these equations alone. Introducing the necessary simplifications in Eqs. (18) of Ref. 6, the equations are

$$m\ddot{\mathbf{R}} + \mathbf{c}_2\bar{\boldsymbol{\phi}}^T\ddot{\mathbf{q}} + 2\Omega_x\mathbf{c}_1\bar{\boldsymbol{\phi}}^T\dot{\mathbf{q}} - (\dot{\Omega}_x\mathbf{c}_1 + \Omega_x^2\mathbf{c}_2)\bar{\boldsymbol{\phi}}^T\mathbf{q} = C^T\mathbf{F} \tag{1a}$$

$$I_C\dot{\Omega}_x + \boldsymbol{\psi}^T\ddot{\mathbf{q}} + \cos\alpha\,\ddot{R}_Y\bar{\boldsymbol{\phi}}^T\mathbf{q} = M \tag{1b}$$

$$M_A\ddot{\mathbf{q}} + \dot{\Omega}_x\boldsymbol{\psi} + (K_A - \Omega_Z^2M_A)\mathbf{q} + \bar{\boldsymbol{\phi}}\mathbf{c}_2^T\ddot{\mathbf{R}} = \mathbf{Q} \tag{1c}$$

where m is the total mass of the spacecraft, $\mathbf{R} = [R_Y \quad R_Z]^T$ is the position vector of the mass center and

$$C = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1^T \\ \mathbf{c}_2^T \end{bmatrix} \tag{2}$$

is the rotation matrix from axes XYZ to axes xyz, in which α is the maneuver angle. Of course, $\Omega_x = \dot{\alpha}$, $\dot{\Omega}_x = \ddot{\alpha}$ are the maneuver angular velocity and acceleration, respectively. In addition, $\mathbf{u}(\mathbf{r},t) = [0 \quad u_z(y,t)]^T$ is the elastic displacement vector, where

$$u_z(y,t) = \boldsymbol{\phi}^T(y)\mathbf{q}(t) \tag{3}$$

in which $\boldsymbol{\phi}(y)$ is a vector of admissible functions and $\mathbf{q}(t)$ is a vector of generalized coordinates. Moreover,

$$\bar{\boldsymbol{\phi}} = \int_{m_A} \boldsymbol{\phi}\, dm_A, \quad \boldsymbol{\psi} = \int_{m_A} y\boldsymbol{\phi}\, dm_A \tag{4a,b}$$

where m_A is the mass of the appendage. Other quantities entering into

Eqs. (1) are the total mass moment of inertia I_c of the undeformed spacecraft about x and the appendage mass and stiffness matrices

$$M_A = \int_{m_A} \underset{\sim}{\phi}(y)\underset{\sim}{\phi}^T(y)\, dm_A, \quad K_A = [\underset{\sim}{\phi},\underset{\sim}{\phi}] \tag{5a,b}$$

where [,] represents an energy inner product (Ref. 8).

Finally, we assume that there are two force actuators F_y and F_z and one torquer M_x acting at the mass center of the rigid body and p torque actuators M_{Ai} acting on the elastic appendage at the points $y = y_i$ ($i = 1,2,\ldots,p$). The torque actuators acting on the appendage can be expressed as a distributed actuator torque in the form

$$m_{Ax} = \sum_{i=1}^{p} M_{Ai}\delta(y - y_i) \tag{6}$$

where $\delta(y - y_i)$ are spatial Dirac delta functions. Then, the forcing terms appearing in Eqs. (1) can be written as follows:

$$\underset{\sim}{F} = F_y\underset{\sim}{j} + F_z\underset{\sim}{k}, \quad M = M_x + \int_{D_A} m_{Ax}\, dD_A = M_x + \sum_{i=1}^{p} M_{Ai} \tag{7a,b}$$

$$\underset{\sim}{Q} = \int_{D_A} m_{Ax}\underset{\sim}{\phi}'\, dD_A = \sum_{i=1}^{p} M_{Ai}\underset{\sim}{\phi}'(y_i) = E\underset{\sim}{M}_A \tag{7c}$$

where $E = [\underset{\sim}{\phi}'(y_1) \ \ \underset{\sim}{\phi}'(y_2) \ \cdots \ \underset{\sim}{\phi}'(y_p)]$ is an $N \times p$ modal participation matrix, in which primes denote derivatives with respect to y, and $\underset{\sim}{M}_A = [M_{A1} \ \ M_{A2} \ \cdots \ M_{Ap}]^T$ is a p-vector of actuator torques acting on the elastic appendage. The reason for using torquers instead of thrusters on the elastic appendgee will become obvious later.

3. The Perturbation Approach to the Maneuvering Problem

We consider the case in which the maneuvering dynamics can be assumed to consist of some large terms associated with the ideal rigid-body maneuvering and some small terms associated with the elastic motions and the perturbations they cause in the rigid-body motions. Consistent with terminology used in perturbation analysis, we refer to the large terms as zero-order terms and to the small terms as first-order terms, and denote them by subscripts 0 and 1, respectively. Hence, we write

$$\underset{\sim}{R} = \underset{\sim}{R}_0 + C_0^T\underset{\sim}{R}_1, \quad \alpha = \alpha_0 + \alpha_1 \tag{8a,b}$$

where $\underset{\sim}{R}_1$ is expressed in terms of components along the body axes, $\underset{\sim}{R}_1 = [R_y \;\; R_z]^T$. Note that $C_0 = C(\alpha_0)$. Moreover, g is regarded as a first-order quantity. It can be verified that

$$\dot{\underset{\sim}{R}} = \dot{\underset{\sim}{R}}_0 + C_0^T(\dot{\underset{\sim}{R}}_1 + \tilde{\Omega}_0\underset{\sim}{R}_1) = \dot{\underset{\sim}{R}}_0 + C_0^T(\dot{\underset{\sim}{R}}_1 + \Omega_0 P\underset{\sim}{R}_1) \tag{9a}$$

$$\ddot{\underset{\sim}{R}} = \ddot{\underset{\sim}{R}}_0 + C_0^T(\ddot{\underset{\sim}{R}}_1 + \dot{\tilde{\Omega}}_0\underset{\sim}{R}_1 + 2\tilde{\Omega}_0\dot{\underset{\sim}{R}}_1 + \tilde{\Omega}_0^2\underset{\sim}{R}_1) = \ddot{\underset{\sim}{R}}_0 + C_0^T[\ddot{\underset{\sim}{R}}_1 + 2\Omega_0 P\dot{\underset{\sim}{R}}_1 + (\dot{\Omega}_0 P - \Omega_0^2 I)\underset{\sim}{R}_1] \tag{9b}$$

$$\Omega_x = \Omega_0 + \dot{\alpha}_1, \quad \dot{\Omega}_x = \dot{\Omega}_0 + \ddot{\alpha}_1 \tag{9c,d}$$

where $\tilde{\Omega}_0 = \Omega_0 P$, in which

$$P = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{10}$$

is a permutation matrix. Similarly, the forcing terms can be divided as follows:

$$\underset{\sim}{F} = \underset{\sim}{F}_0 + \underset{\sim}{F}_1, \quad M = M_0 + M_1 \tag{11a,b}$$

whereas $\underset{\sim}{Q}$ is assumed to be of first order.

Inserting Eqs. (8)-(11) into Eqs. (1) and neglecting second-order terms in the perturbations, we obtain the zero-order equations

$$m\ddot{\underset{\sim}{R}}_0 = C_0^T\underset{\sim}{F}_0, \quad I_c\dot{\Omega}_0 = M_0 \tag{12a,b}$$

where, considering Eqs. (7), $\underset{\sim}{F}_0 = F_{y0}\underset{\sim}{j} + F_{z0}\underset{\sim}{k}$, $M_0 = M_{x0}$. From Eqs. (12), it is obvious that, owing to the fact that the motion is referred to the mass center C, the translational and rotational equations are independent of each other. Because the interest lies in a slewing maneuver, there is no loss of generality in assuming that $\underset{\sim}{R}_0 = \dot{\underset{\sim}{R}}_0 = \ddot{\underset{\sim}{R}}_0 = \underset{\sim}{0}$. This permits us to dispense with Eq. (12a), so that we are left with a single zero-order equation, Eq. (12b). Moreover, we obtain the first-order equations, which can be written in the compact form

$$M\ddot{\underset{\sim}{x}} + G\dot{\underset{\sim}{x}} + K\underset{\sim}{x} = \underset{\sim}{X} \tag{13}$$

where

$$\underset{\sim}{x} = [\underset{\sim}{R}_1^T \;\; \alpha_1 \;\; g^T]^T, \quad \underset{\sim}{X} = \begin{bmatrix} \underset{\sim}{F}_1 \\ M_1 \\ \underset{\sim}{Q} - \dot{\Omega}_0\underset{\sim}{\psi} \end{bmatrix} \tag{14a,b}$$

are (3+N)-dimensional displacement and generalized force vectors, respectively, in which

$$\underset{\sim}{F}_1 = F_{y1}\underset{\sim}{j} + F_{z1}\underset{\sim}{k}, \quad M_1 = M_{x1} + \sum_{i=1}^{p} M_{Ai} \tag{15a,b}$$

and $\underset{\sim}{Q}$ is given by Eq. (7c), and

$$M = \begin{bmatrix} mI & 0 & \underset{\sim}{e}_2\bar{\underset{\sim}{\phi}}^T \\ 0 & I_C & \underset{\sim}{\psi}^T \\ \bar{\underset{\sim}{\phi}}\underset{\sim}{e}_2^T & \underset{\sim}{\psi} & M_A \end{bmatrix}, \quad G = \begin{bmatrix} 2m\Omega_0 P & 0 & -2\Omega_0\underset{\sim}{e}_1\bar{\underset{\sim}{\phi}}^T \\ 0 & 0 & 0 \\ 2\Omega_0\bar{\underset{\sim}{\phi}}\underset{\sim}{e}_1^T & 0 & 0 \end{bmatrix} \tag{16a,b}$$

$$K = \begin{bmatrix} m(\dot{\Omega}_0 P - \Omega_0^2 I) & 0 & -(\dot{\Omega}_0\underset{\sim}{e}_1 + \Omega_0^2\underset{\sim}{e}_2)\bar{\underset{\sim}{\phi}}^T \\ 0 & 0 & 0 \\ \bar{\underset{\sim}{\phi}}(\dot{\Omega}_0\underset{\sim}{e}_1^T - \Omega_0^2\underset{\sim}{e}_2^T) & 0 & K_A - \Omega_0^2 M_A \end{bmatrix} \tag{16c}$$

are coefficient matrices, where $\underset{\sim}{e}_1 = [1 \quad 0]^T$ and $\underset{\sim}{e}_2 = [0 \quad 1]^T$ are standard unit vectors. Note that G and K depend on time through Ω_0 and $\dot{\Omega}_0$.

It will prove convenient to express the generalized force vector in terms of actual actuator forces and torques. To this end, we introduce the notation

$$F_{y1} = T_1, \; F_{z1} = T_2, \; M_{x1} = T_3, \; M_{Ai} = T_{3+i}, \quad i = 1,2,\ldots,p \tag{17a}$$

$$D = \left[\begin{array}{c|cccc} & 0 & 0 & \ldots & 0 \\ I & 0 & 0 & \ldots & 0 \\ & 1 & 1 & \ldots & 1 \\ \hline 0 & & & E & \end{array}\right], \quad \underset{\sim}{\Psi} = [\underset{\sim}{0}^T \quad \underset{\sim}{\psi}^T]^T \tag{17b,c}$$

and consider Eqs. (7c) and (15) to obtain

$$\underset{\sim}{X} = D\underset{\sim}{T} - \dot{\Omega}_0\underset{\sim}{\Psi} \tag{18}$$

where $\underset{\sim}{T}$ is the (3+p)-vector of actuator forces. Moreover, we wish to express the matrices G and K in the form

$$G = 2\Omega_0 G^*, \quad K = K_c + \dot{\Omega}_0 G^* - \Omega_0^2 K_s \tag{19a,b}$$

where

$$G^* = \begin{bmatrix} mP & 0 & -\underline{e}_1 \bar{\underline{\phi}}^T \\ 0 & 0 & 0 \\ \bar{\underline{\phi}} \underline{e}_1^T & 0 & 0 \end{bmatrix} = -(G^*)^T \tag{20a}$$

$$K_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K_A \end{bmatrix} = K_c^T, \quad K_s = \begin{bmatrix} mI & 0 & \underline{e}_2 \bar{\underline{\phi}}^T \\ 0 & 0 & 0 \\ \bar{\underline{\phi}} \underline{e}_2^T & 0 & M_A \end{bmatrix} = K_s^T \tag{20b,c}$$

and we recognize that G^* is skew symmetric, $\overline{K}$ is the constant part of K, a symmetric matrix, and K_s is symmetric as well.

4. Pseudo-Modal Equations of Motion

Equation (13) represents a set of 3+N linear ordinary differential equations with time-dependent coefficients. For future reference, we wish to rewrite the equations in a different form. To this end, we observe that by letting $\Omega_0 = \dot{\Omega}_0 = 0$ Eq. (13) reduces to a time-invariant set of equations. The eigenvalue problem for the time-invariant system has the form $K_c U = MU\Lambda$, where U is a $(3+N)\times(3+N)$ matrix of eigenvectors and $\Lambda = \mathrm{diag}[0 \;\; 0 \;\; 0 \;\; \omega_1^2 \;\; \omega_2^2 \; \ldots \; \omega_N^2]$ is a diagonal matrix of eigenvalues, in which $\omega_i (i = 1,2,\ldots,N)$ are recognized as the natural frequencies of the nonmaneuvering spacecraft. Because M and K_c are real and symmetric and, moreover, M is positive definite and K_c is positive semidefinite, the eigenvectors are real and orthogonal with respect to M and K_c and the nonzero eigenvalues are real and positive (Ref. 2). The eigenvectors can be normalized so as to satisfy $U^T M U = I$, $U^T K_c U = \Lambda$. It can also be verified that

$$U^T G^* U = \overline{G}, \quad U^T K_s U = \overline{K} \tag{21a,b}$$

where

$$\overline{G} = \left[\begin{array}{ccc|c} 0 & -1 & 0 & \\ 1 & 0 & 0 & \underline{0} \\ 0 & 0 & 0 & \\ \hline & \underline{0}^T & & 0 \end{array}\right], \quad \overline{K} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underline{0} \\ 0 & 0 & 0 & \\ \hline & \underline{0}^T & & \overline{K}_{22} \end{array}\right] \tag{22a,b}$$

in which

$$\bar{K}_{22} = I_N + \frac{1}{I_c} U_{22}^T \underset{\sim}{\psi}^T \underset{\sim}{\psi} U_{22} \tag{23}$$

and we note that I_N is the N×N identity matrix and U_{22}^T the N×N lower right submatrix of U.

Next, let us introduce the linear transformation

$$\underset{\sim}{x}(t) = U\underset{\sim}{v}(t) \tag{24}$$

into Eq. (13), multiply on the left by U^T and obtain

$$\ddot{\underset{\sim}{v}} + 2\Omega_0 \bar{G}\dot{\underset{\sim}{v}} + (\Lambda + \dot{\Omega}_0\bar{G} - \Omega_0^2\bar{K})\underset{\sim}{v} = \underset{\sim}{V} \tag{25}$$

where

$$\underset{\sim}{V} = U^T\underset{\sim}{X} = U^T D\underset{\sim}{T} - \dot{\Omega}_0 U^T \underset{\sim}{\Psi} \tag{26}$$

Equation (25) represents a set of pseudo-modal equations.

Due to the nature of the coefficient matrices, $\bar{G}$, Λ and $\bar{K}$, as well of the excitation vector $\underset{\sim}{V}$, the pseduo-modal equations can be separated into

$$\ddot{\underset{\sim}{v}}_T + 2\Omega_0 P\dot{\underset{\sim}{v}}_T + (\dot{\Omega}_0 P - \Omega_0^2 I)\underset{\sim}{v}_T = \underset{\sim}{V}_T = [T_1 \quad T_2]^T \tag{27a}$$

$$\ddot{v}_R = V_R = \sum_{j=1}^{p+1} T_j \tag{27b}$$

$$\ddot{\underset{\sim}{v}}_E + (\Lambda_E - \Omega_0^2\bar{K}_{22})\underset{\sim}{v}_E = \underset{\sim}{V}_E = U_{22}^T E\underset{\sim}{T}_E - \dot{\Omega}_0 U_{22}^T \underset{\sim}{\psi} \tag{27c}$$

where $\underset{\sim}{v}_T = [v_1 \quad v_2]^T$ is a vector of perturbations in the rigid-body translations, $v_R = v_3$ is the perturbation in the rigid body rotation, $\underset{\sim}{v}_E = [v_4 \quad v_5 \ \dots \ v_{3+N}]^T$ is a vector of pseudo-modal coordinates corresponding to elastic motions, $\Lambda_E = \text{diag}\ (\omega_1^2 \quad \omega_2^2 \ \dots \ \omega_N^2)$ and $T_E = [T_4 \quad T_5 \ \dots \ T_{3+p}]$ is a vector of torquers on the elastic appendage. Equations (27) indicate that the rigid-body translations, rigid-body rotations and elastic motions can be treated independently. This justifies the choice of actuators in Sec. 2.

5. General Control Policy

The control can be divided into two tasks to be carried out simultaneoulsy. The first task consists of the rigid-body slewing of the structure according to Eq. (12b). The second task consists of designing a regulator to suppress the perturbations from the rigid-body

slewing. To this end, we use Eqs. (27). Of course, the actuator forces and torques represent the controls for both tasks.

i. Rigid-body slewing.

For the rigid-body slewing, we use a minimum-time control policy, which implies bang-bang control (Ref. 9). Assuming an ideal actuator and recognizing that $\Omega_0 = \dot{\alpha}_0$, where α_0 is the rigid-body slewing angle, Eq. (12b) represents a linear time-invariant system with two real eigenvalues. As a result, there is only one switching time, namely halfway through the maneuvering time interval. Hence, denoting by t_0 the initial and by t_f the final time, the switching time is simply $t_1 = (t_f - t_0)/2$, so that the control law is

$$\dot{\Omega}_0 = \begin{cases} c \text{ for } t_0 \le t \le t_1 \\ -c \text{ for } t_1 < t \le t_f \end{cases} \tag{28}$$

Of course, the torque is $M_0 = I_c\dot{\Omega}_0$, so that c must be such that the actuator operates at the saturation point. From Eq. (28), we conclude that the slewing angular velocity is

$$\Omega_0 = \begin{cases} ct \text{ for } t_0 \le t \le t_1 \\ -c(t - 2t_1) \text{ for } t_1 < t \le t_f \end{cases} \tag{29}$$

and the slewing angle is

$$\alpha_0 = \begin{cases} \frac{1}{2} ct^2 \text{ for } t_0 \le t \le t_f \\ -ct_1^2 - c(\frac{1}{2} t^2 - 2t_1 t) \text{ for } t_1 < t \le t_f \end{cases} \tag{30}$$

and we note that Eqs. (29) and (30) imply that $t_0 = 0$.

ii. Perturbation suppression

The perturbations are governed by Eqs. (27) and they represent a time-varving system subjected to persistent disturbances generated by the inertial forces resulting from the rigid-body slewing. These perturbations are to be suppressed during the maneuver, i.e., during the time time interval $t_f - t_0$. The time-dependent functions in Eqs. (27) are $\Omega_0(t)$ and $\dot{\Omega}_0(t)$, which represent the commanded angular velocity and acceleration of the "rigid body" slewing. Hence, they are both known a priori.

Another aspect of the problem is the high dimensionality. Due to the on-line computer limitations, as well as inaccuracies in the model higher states, the originally truncated model, Eq. (27c), must be truncated again. The newly truncated model, referred to as the reduced-order model (ROM), will be controlled by an optimal reduced-order controller (ROC), which will accommodate disturbances during maneuver.

The truncation mentioned above is an open-loop truncation in which the higher states are ignored on the assumption that their excitation during maneuver is minimal. Moreover, any damping inherent in the system tends to damp out these states the fastest. The validity of this assumption can be verified by simulations.

The drawbacks of controlling a distributed-parameter system by a ROC are the well-known control and observation spillover (Refs. 10 and 11). We assume in this paper that observation spillover effects are mitigated by using a sufficient number of sensors, i.e., 2 translational displacement, 1+N angular displacement, 2 translational velocity and 1+N angular velocity sensors, where the sensors are placed so as to permit an accurate estimate of the controlled state vector.

As pointed out in Sec. 4, the rigid-body translations, the rigid-body rotation and the elastic coordinates in the pseudo-modal equations, Eqs. (27), can be treated independently. In particular, we first design controls for the elastic perturbations. Then, we design the control for the rotational perturbation, taking into account the fact that the actuators for the elastic motion perturb the rotational motion, as can be concluded from Eq. (27b). The rigid-body translations are entirely decoupled from the rotational and elastic perturbations. Concentrating on the elastic motions, we can introduce the elastic state vector $\underset{\sim}{z}(t) = [\underset{\sim}{v}^T(t) \,\vdots\, \dot{\underset{\sim}{v}}^T(t)]^T = [\underset{\sim}{z}_C^T(t) \,\vdots\, \underset{\sim}{z}_U^T(t)]^T$, where $\underset{\sim}{z}_C(t) = [\underset{\sim}{v}_C^T(t) \,\vdots\, \dot{\underset{\sim}{v}}_C^T(t)]$ is the controlled part of the state vector and $\underset{\sim}{z}_U(t) = [\underset{\sim}{v}_U^T(t) \,\vdots\, \dot{\underset{\sim}{v}}_U^T]^T$ is the uncontrolled part. Then, considering Eq. (27c), the state equations can be written in the form

$$\begin{bmatrix} \dot{\underset{\sim}{z}}_C \\ \dot{\underset{\sim}{z}}_U \end{bmatrix} = \left[\begin{array}{c:c} A_C & A_{CU} \\ \hdashline A_{UC} & A_U \end{array}\right] \begin{bmatrix} \underset{\sim}{z}_C \\ \underset{\sim}{z}_U \end{bmatrix} + \begin{bmatrix} B_C \\ B_U \end{bmatrix} \underset{\sim}{T}_E - \dot{\Omega}_0 \begin{bmatrix} \underset{\sim}{R}_C \\ \underset{\sim}{R}_U \end{bmatrix} \tag{31}$$

in which

$$A_C = \left[\begin{array}{c:c} 0 & I_C \\ \hdashline -\Lambda_C + \Omega_0^2 \bar{K}_C & 0 \end{array}\right], \quad A_{CU} = \left[\begin{array}{c:c} 0 & 0 \\ \hdashline \Omega_0^2 \bar{K}_{CU} & 0 \end{array}\right] \tag{32a,b}$$

$$A_{UC} = \left[\begin{array}{c|c} 0 & 0 \\ \hline \Omega_0^2 \bar{K}_{UC} & 0 \end{array}\right], \quad A_U = \left[\begin{array}{c|c} 0 & I_U \\ \hline -\Lambda_U + \Omega_0^2 \bar{K}_U & 0 \end{array}\right] \tag{32c,d}$$

$$B_C = \left[\begin{array}{c} 0 \\ \hline E_C \end{array}\right], \quad B_U = \left[\begin{array}{c} 0 \\ \hline E_U \end{array}\right], \quad \underset{\sim}{R}_C = \left[\begin{array}{c} \underset{\sim}{0} \\ \hline \underset{\sim}{\psi}_C \end{array}\right], \quad \underset{\sim}{R}_U = \left[\begin{array}{c} \underset{\sim}{0} \\ \hline \underset{\sim}{\psi}_U \end{array}\right] \tag{32e-h}$$

where we introduced the notation $U_{22C}^T E = E_C$, $U_{22U}^T E = E_U$, $U_{22C}^T \underset{\sim}{\psi} = \underset{\sim}{\psi}_C$, $U_{22U}^T \underset{\sim}{\psi} = \underset{\sim}{\psi}_U$, and we note that the dimensions of the various submatrices are consistent with the dimensions of $\underset{\sim}{z}_C$ and $\underset{\sim}{z}_U$, which are $2N_C$ and $2N_U$, respectively. Of course, dim $\underset{\sim}{z}_C$ + dim $\underset{\sim}{z}_U = 2N$.

6. Control of the Elastic Reduced-Order Model (ROM)

The elastic ROM is obtained from Eq. (31) by ignoring the uncontrolled state vector $\underset{\sim}{z}_U$ and neglecting the coupling matrix A_{CU}. Hence, we propose to retain for control the reduced-order model

$$\dot{\underset{\sim}{z}}_C = A_C \underset{\sim}{z}_C + B_C \underset{\sim}{T}_E - \dot{\Omega}_0 \underset{\sim}{R}_C, \quad \underset{\sim}{y}_C = M_C \underset{\sim}{z}_C \tag{33a,b}$$

where the matrices A_C and B_C and the vector $\underset{\sim}{R}_C$ are given by Eqs. (32a), (32e) and (32g), respectively, and $\dot{\Omega}_0$ is given by Eq. (28), in which c is a given constant. Moreover, $\underset{\sim}{y}_C$ is the output vector and it represents the quantity to be controlled. In our case, the output vector has two components, namely, the angular deflection and velocity at various locations $y_1, y_2, \ldots, y_\ell$ of the beam. Hence, considering Eqs. (3), (14a), (24) and (27), and recalling the definition of $\underset{\sim}{z}_C$, we have

$$M_C = \begin{bmatrix} F^T U_{22} \\ F^T U_{22} \end{bmatrix} \tag{34}$$

where $F = [\underset{\sim}{\phi}'(y_1) \quad \underset{\sim}{\phi}'(y_2) \ \ldots \ \underset{\sim}{\phi}'(y_\ell)]$ is an $\ell \times 2N_C$ matrix, in which primes denote derivatives with respect to y.

The object of the control is to drive $\underset{\sim}{y}_C$ to zero by the end of the maneuver, or

$$\lim_{t \to t_f} \underset{\sim}{y}_C(t) = \underset{\sim}{0} \tag{35}$$

Because of the persistent disturbance $-\dot{\Omega}_0 \underset{\sim}{R}_C$, for Eq. (35) to be satisfied optimally, we must also have

$$\lim_{t \to t_f} B_C \underset{\sim}{T}_E(t) = c\underset{\sim}{R}_C \tag{36}$$

For Eq. (35) to be satisfied, the range space of $\underset{\sim}{R}_C$ must be contained in the range space of B_C. Hence, because the order of the ROM is equal to the dimension of $\underset{\sim}{R}_C$, the number of actuators must be equal to one half the order of the ROM, or $p = N_C$. Of course, the location of the actuators must be such that $E_C = U_{22C}^T E$ has full rank.

The problem of disturbance accommodation using an LQR was discussed in Refs. 12-17, among others. In view of the dimension of $\underset{\sim}{R}_C$, as well as our goal of satisfying Eq. (35) optimally, the approach suggested in Refs. 12 and 13 appears suitable. Moreover, to force $\underset{\sim}{y}_C(t)$ to zero within the time interval (t_0, t_f), we include an exponential term in the performance index. This term is equivalent to prescribing a degree of stability to an infinite final time, time-invariant problem, as suggested in Ref. 17. In the Appendix, it is shown that this exponential term does provide the minimal rate of convergence even in time-varying systems.

Let us consider a new variable vector $\underset{\sim}{u}(t)$ defined by

$$B_C \underset{\sim}{u}(t) = B_C \underset{\sim}{T}_E(t) + \underset{\sim}{F}_C(t) \tag{37}$$

where

$$\underset{\sim}{F}_C = \begin{cases} -c\underset{\sim}{R}_C & \text{for } t_0 \le t \le t_1 \\ c\underset{\sim}{R}_C & \text{for } t_1 < t \le t_f \end{cases} \tag{38}$$

and introduce the new state vector $\underset{\sim}{x}_C^T(t) = [\underset{\sim}{z}_C^T(t) \;\; \underset{\sim}{u}^T(t)]^T$. Then, the new state equations can be written in the form

$$\dot{\underset{\sim}{x}}_C(t) = H(t)\underset{\sim}{x}_C(t) + G\underset{\sim}{u}_1(t), \quad \underset{\sim}{u}_1(t) = \dot{\underset{\sim}{u}}(t) \tag{39a,b}$$

in which

$$H = \left[\begin{array}{c|c} A_C & B_C \\ \hline 0 & 0 \end{array}\right], \quad G = \left[\begin{array}{c} 0 \\ \hline I \end{array}\right] \tag{40a,b}$$

We propose to derive an optimal control by minimizing the performance measure

$$J = \underset{\sim}{x}_C^T(T_f) H \underset{\sim}{x}_C(T_f) + \int_{T_0}^{T_f} e^{2\alpha t}(\underset{\sim}{x}_C^T Q_1 \underset{\sim}{x}_C + \underset{\sim}{u}_1^T S \underset{\sim}{u}_1)\, dt \tag{41}$$

where

$$Q_1 = \left[\begin{array}{c|c} Q & 0 \\ \hline 0 & R \end{array}\right], \quad Q = M_C^T Q_0 M_C \tag{42a,b}$$

The matrices Q, R and S are real symmetric and positive definite. Moreover, T_0 and T_f are equal to t_0 and t_1 for the first stage and to t_1+ and t_f for the second stage.

The formulation given by Eqs. (39) and (41) can be transformed into a standard LQR formulation by defining the new state and control vectors

$$\hat{\underset{\sim}{x}}_C(t) = e^{\alpha t}\underset{\sim}{x}_C(t), \quad \hat{\underset{\sim}{u}}_1(t) = e^{\alpha t}\underset{\sim}{u}_1(t) \tag{43a,b}$$

In terms of the newly defined vectors, the state equations have the form

$$\dot{\hat{\underset{\sim}{x}}}_C(t) = \hat{H}(t)\hat{\underset{\sim}{x}}_C(t) + G\hat{\underset{\sim}{u}}_1(t), \quad \hat{\underset{\sim}{x}}_C(T_0) = e^{\alpha T_0}\underset{\sim}{x}_C(T_0) \tag{44a,b}$$

where $\hat{H}(t) = H(t) + \alpha I$. Similarly, the modified performance measure is

$$J_\alpha = \hat{\underset{\sim}{x}}_C^T(T_f)\hat{H}\hat{\underset{\sim}{x}}_C(T_f) + \int_{T_0}^{T_f}(\hat{\underset{\sim}{x}}_C^T Q_1 \hat{\underset{\sim}{x}}_C + \hat{\underset{\sim}{u}}_1^T S \hat{\underset{\sim}{u}}_1)dt \tag{45}$$

Inserting Eqs. (43) into Eq. (45), it is easy to verify that $J_\alpha = J$.

The optimal control law can be shown to have the form

$$\underset{\sim}{u}_1(t) = \dot{\underset{\sim}{u}}(t) = -S^{-1}G^T P(t)\underset{\sim}{x}_C(t) \tag{46}$$

where P(t) is the solution of the Riccati equation

$$\dot{P}(t) = -P(t)[H(t) + \alpha I] - [H^T(t) + \alpha I]P(t) + P(t)GS^{-1}GP(t) - Q_1 \tag{47}$$

and is subject to $P(T_f) = \hat{H}$. We can partition the Riccati matrix as follows:

$$P = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{12}^T & P_{22} \end{array}\right] \tag{48}$$

where P_{11} is $2N_C \times 2N_C$, P_{12} is $2N_C \times N_C$ and P_{22} is $N_C \times N_C$. Then, Eq. (47) can be rewritten in terms of the original system, Eqs. (39) and (41), in the form

$$\dot{P}_{11} = -P_{11}A_C - A_C^T P_{11} - 2\alpha P_{11} + P_{12}S^{-1}P_{12}^T - Q \tag{49a}$$

$$\dot{P}_{12} = -P_{11}B_C - A_C^T P_{12} - 2\alpha P_{12} + P_{12}S^{-1}P_{22} \tag{49b}$$

$$\dot{P}_{22} = -P_{12}^T B_C - B_C^T P_{12} - 2\alpha P_{22} + P_{22}S^{-1}P_{22} - R \tag{49c}$$

where the submatrices of P are subject to $P_{ij}(t_f) = \hat{H}_{ij}(i,j = 1,2)$.

Next, we wish to express the actuator vector $\mathbf{T}_E(t)$ in terms of the pseudo-modal coordinates. Inserting Eqs. (41b) and (48) into Eq. (46), we obtain

$$\dot{\mathbf{u}}(t) = -S^{-1}P_{12}^T(t)\mathbf{z}_C(t) - S^{-1}P_{22}(t)\mathbf{u}(t) \tag{50}$$

But, $\mathbf{u}(t)$ is related to the actuator vector via Eq. (37). Hence, using Eqs. (33a) and (37), recalling Eqs. (28) and recognizing that $\mathbf{F}_C(t) = \mathbf{0}$ over each control stage, we conclude that the actuator vector satisfies

$$\dot{\mathbf{T}}_E(t) = -S^{-1}[P_{12}^T(t) - P_{22}(t)B_C^\dagger A_C(t)]\mathbf{z}_C - S^{-1}P_{22}(t)B_C^\dagger\dot{\mathbf{z}}_C(t) \tag{51}$$

where $B_C^\dagger = (B_C^TB_C)^{-1}B_C^T = [0 \mid (E_C^TE_C)^{-1}E_C^T] = [0 \mid E_C^\dagger]$, in which the dagger denotes the pseudo-inverse of a matrix. Introducing the notation

$$S^{-1}[P_{12}^T(t) - P_{22}(t)B_C^\dagger A_C(t)] = [K_1(t) \mid K_2(t)] \tag{52a}$$

$$S^{-1}P_{22}(t)B_C^\dagger = S^{-1}P_{22}(t)[0 \mid E_C^\dagger] = [0 \mid K_3(t)] \tag{52b}$$

the actuator vector can be written in the form

$$\mathbf{T}_E(t) = \mathbf{T}_E(T_0) - \int_{T_0}^{t}[K_1(\tau)\mathbf{v}_C(\tau) + K_2(\tau)\dot{\mathbf{v}}_C(\tau) + K_3(\tau)\ddot{\mathbf{v}}_C(\tau)]d\tau \tag{53}$$

where T_0 is either t_0 or t_1, depending on the control stage. For convenience, we take $\mathbf{T}_E(T_0) = \mathbf{0}$.

It should be pointed out that the above integral feedback control is capable of compensating for constant unknown disturbances, such as might occur when the vector $\boldsymbol{\psi}$ in the disturbance vector $-\dot{\Omega}_0\boldsymbol{\psi}$ is not known exactly.

7. Response of the Full (Discretized) Model

We refer to the discretized model given by Eq. (31) as full, although it represents a truncated model relative to the distributed system. The idea is that the states ignored in arriving at Eq. (31) do not participate anyway. The closed-loop system is obtained by introducing Eq. (53) into Eq. (31). Because the feedback control law is in integral form, we will find it convenient to introduce the expanded 3N-state vector given by $\mathbf{w} = [\mathbf{w}_C^T \mid \mathbf{w}_U^T]^T = [\mathbf{v}_C^T \mid \dot{\mathbf{v}}_C^T \mid \ddot{\mathbf{v}}_C^T \mid \mathbf{v}_U^T \mid \dot{\mathbf{v}}_U^T \mid \ddot{\mathbf{v}}_U^T]^T$. Then, using Eqs. (31) and (53), we obtain the closed-loop equation

$$\dot{\mathbf{w}}(t) = \bar{A}(t)\mathbf{w}(t) \tag{54}$$

where

$$\bar{A} = \left[\begin{array}{c|c} \bar{A}_C & \bar{A}_{CU} \\ \hline \bar{A}_{UC} & \bar{A}_U \end{array}\right] \tag{55}$$

in which

$$\bar{A}_C = \left[\begin{array}{c|c|c} 0 & I_C & 0 \\ \hline 0 & 0 & I_C \\ \hline 2\Omega_0\dot{\Omega}_0\bar{K}_C - E_C K_1 & \Omega_0^2\bar{K}_C - \Lambda_C - E_C K_2 & -E_C K_3 \end{array}\right] \tag{56a}$$

$$\bar{A}_{CU} = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 2\Omega_0\dot{\Omega}_0\bar{K}_{CU} & \Omega_0^2\bar{K}_{CU} & 0 \end{array}\right] \tag{56b}$$

$$\bar{A}_{UC} = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 2\Omega_0\dot{\Omega}_0\bar{K}_{UC} - E_U K_1 & \Omega_0^2\bar{K}_{UC} - E_U K_2 & -E_U K_3 \end{array}\right] \tag{56c}$$

$$\bar{A}_U = \left[\begin{array}{c|c|c} 0 & I_U & 0 \\ \hline 0 & 0 & I_U \\ \hline 2\Omega_0\dot{\Omega}_0\bar{K}_U & \Omega_0^2\bar{K}_U - \Lambda_U & 0 \end{array}\right] \tag{56d}$$

where we regarded $\ddot{\Omega}_0$ as zero.

In many structures, the off-diagonal entries of the matrix $\bar{K}_{22}$, Eq. (23), are very small compared to one. This is particularly true in the case under investigation. In such cases, the coupling matrix $\bar{A}_{CU}$, Eq. (56b), is very small and can be ignored without affecting the results very much. Under these circumstances, we conclude from Eqs. (54) and (55) that the uncontrolled states do not affect the controlled states. On the other hand, because $\bar{A}_{UC} \neq 0$, and the terms preventing $\bar{A}_{UC}$ from being zero can be traced to the feedback control, we conclude that the system is subjected to control spillover. It is assumed here that the controlled states are fully observable. Hence, the control spillover cannot lead to instability, although significant performance degradation can occur if the number of controlled modes is too small.

8. Numerical Example

The structure considered is as shown in Fig. 1, in which the length of the beam is L = 24 ft. Moreover, the first seven natural frequencies of vibration in the yz-plane are 0.32, 1.8, 5.0, 9.8, 16.2, 24.2 and 33.8 (Hz). A damping factor of 0.01 is included.

Two different maneuvers were considered. The first consisted of a 180° slew in 7.05s and the second of 20° slew in 3.98s.

The ROM was assumed to have four flexible degrees of freedom and the originally discretized system was assumed to have seven flexible degrees of freedom. There were seven actuators, three rigid-body actuators mounted on the hub and four actuators mounted on the beam at the points $y_i = iL/4$ $(i = 1,2,3,4)$. The values of the control parameters were taken as $Q = R = I$, $S = 0.1\ I$, $\alpha = 5.0$. Control is terminated at t_f.

The rigid-body maneuver is shown in Fig. 2. It is an ideal minimum-time maneuver. Figure 3 shows the moment output of the actuators on the beam. Every actuator generates a near-constant moment, compensating for the constant disturbance. The effect of the integral feedback is obvious. Figure 4-7 show the angular error (measured in degrees) at the tip of the beam, at which point the deflection is the largest. The error represents the difference between the actual displacement and the rigid-body displacement, where the latter can be obtained from Fig. 2. The characteristics of the error for both maneuvers are similar, although the first maneuver, being faster, exhibits a larger error. Figures 4 and 6 show the error of the controlled model alone. Convergence to zero (10^{-10} degrees) is achieved during the control interval (T_0, T_f) by choosing a proper convergence factor α. The "full model" experienced an error due to control spillover into the uncontrolled states, as well as to uncompensated disturbances. This error is shown in Figs. 5 and 7. The inherent damping in the system causes this error to decay after the termination of the control.

9. Conclusions

For maneuvers characterized by angular velocities not much higer than the lowest natural frequency of the beam, the time-varying terms in $\bar{A}_C(t)$ tend to be negligible in the case of closed-loop control. In such cases, the solution of the Riccati equation requires less computational effort, as $\bar{A}_C$ does not need updating. Note, however, that the Riccati matrix still depends on time.

The errors result mainly from control spillover into the uncontrolled states. The contributions to the errors are primarily from the lowest uncontrolled states. The error can be reduced by increasing the

number of controlled modes. Even for the model used here, the error is not very large. Indeed, the error in slope at the tip of the beam drops to less than 0.1° at $t = 1.1\ t_f$. This good performance can be attributed to the large separation in the natural frequencies, which is typical of one-dimensional structures such as beams. For two- and three-dimensional structures, the natural frequencies tend to be spaced more closely, so that the error is likely to increase.

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References

1. Junkins, J. L. and Turner, J. D., Optimal Spacecraft Rotational Maneuvers, Elsevier, New York, 1986.

2. Turner, J. D. and Chun, H. M., "Optimal Distributed Control of a Flexible Spacecraft During a Large-Angle Maneuver," Journal of Guidance, Control, and Dynamics, Vol. 7, No. 3, 1984, pp. 257-264.

3. Chun, H. M., Turner, J. D. and Juang, J. N., "Frequency-Shaped Large Angle Maneuver," Paper No. AIAA-87-0174, AIAA 25th Aerospace Sciences Meeting, Reno, NV, January, 12-15, 1987.

4. Ben-Asher, J., Burns, J. A. and Cliff, E. M., "Time Optimal Slewing of Flexible Spacecraft," 26th IEEE Conference on Decision and Control, Los Angeles, CA, December 9-11, 1987.

5. Thompson, R. C., Junkins, J. L. and Vadali, S. R., "Near Minimum Time, Open Loop Control of Dynamic Systems," Paper No. AIAA-87-0958-CP, AIAA Dynamics Specialists Conference, Monterey, CA, April 9-10, 1987.

6. Meirovitch, L. and Quinn, R. D., "Equations of Motion for Maneuvering Flexible Spacecraft," Journal of Guidance, Control, and Dynamics, Vol. 10, No. 5, 1987, pp. 453-465.

7. Meirovitch, L. and Quinn R. D., "Maneuvering and Vibration Control of Flexible Spacecraft," Journal of Astronautical Sciences, Vol. 35, No. 3, 1987.

8. Meirovitch, L., Computational Methods in Structural Dynamics, Sijthoff & Noordhoff, The Netherlands, 1980.

9. Kirk, D. E., Optimal Control Theory, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1970.

10. Balas, M. J., "Active Control of Flexible Systems," Journal of Optimization Theory and Applications, Vol. 25, No. 3, 1978, pp. 415-436.

11. Meirovitch, L. and Baruh, H., "On the Problem of Observation Spillover in Self-Adjoint Distributed-Parameter Systems," Journal of Optimization Theory and Applications, Vol. 39, No. 2, 1983, pp. 269-291.

12. Johnson, C. D., "Optimal Control of the Linear Regulator with Constant Disturbances," IEEE Transactions on Automatic Control, Vol. AC-13, No. 4, 1968, pp. 416-421.

13. Anderson, B. D. O. and Moore, J. B., Linear Optimal Control, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1971.

14. Wong, K. P. and Seborg, D. E., "Optimal Proportional Plus Integral Control for Regulator and Tracking Problems," Optimal Control Applications and Methods, Vol. 6, 1985, pp. 335-350.

15. Balchen, J. G., Endresen, T., Fjeld, M. and Olsen, T. O., "Multivariable PID Estimation and Control in Systems with Biased Disturbances," Automatica, Vol. 9, 1973, pp. 295-307.

16. Johnson, C. D., "Accommodation of External Disturbances in Linear Regulator and Servomechanism Problems," IEEE Transactions on Automatic Control, Vol. AC-16, No. 6, 1971, pp. 635-644.

17. Anderson, B. D. O. and Moore, J. B., "Linear System Optimization with Prescribed Degree of Stability," Proceedings of the IEE, Vol. 116, No. 12, 1969, pp. 2083-2087.

18. Kwakernaak, H. and Sivan, R., Linear Optimal Control Systems, Wiley-Interscience, New York, 1972.

19. Weiss, L. and Infante, E. F., "On the Stability of Systems Defined Over a Finite Time Interval," Proceedings of the National Academy of Sciences, Vol. 54, July 1965, pp. 44-48.

20. Heinen, J. A. and Wu, S. H., "Further Results Concerning Finite Time Stability," IEEE Transactions on Automatic Control, Vol. AC-14, No. 4, 1969, pp. 211-212.

Appendix

The purpose of this Appendix is to examine the convergence rate of the controlled state $\underset{\sim}{x}_C(t)$. To this end, we recall that the optimal control law for the system described by Eqs. (44) and subject to the performance index (45) is given by Eq. (46), in which P(t) is the

solution of the matrix Riccati equation, Eq. (47). The matrix $P(t)$ is symmetri_ and positive definite for $t \epsilon [t_0, t_f)$.

To help with the developments, we introduce the following:
Definition (Ref. 19): The system described by Eqs. (44) is finite-time stable (FTS) in the interval $T = [t_0, t_f)$ with respect to $(\gamma, \beta, t_0, \| \ \|)$ for every trajectory $\hat{\underset{\sim}{x}}_C(t)$ if $\|\hat{\underset{\sim}{x}}_C(t_0)\| < \gamma$ implies $\|\hat{\underset{\sim}{x}}_C(t)\| < \beta$ for every $t \epsilon T$, where $\| \ \|$ denotes the Euclidean norm.

Moreover, we introduce the notation

$$B(a) = \{\hat{\underset{\sim}{x}}_C;\ \|\hat{\underset{\sim}{x}}_C\| < a\},\quad V_m^a(t) = \min_{\|\hat{\underset{\sim}{x}}_C\| = a} V(\hat{\underset{\sim}{x}}_C, t) \tag{A1,A2}$$

$$V_M^a(t) = \max_{\|\hat{\underset{\sim}{x}}_C\| = a} V(\hat{\underset{\sim}{x}}_C, t),\quad V_M^B(t) = \sup_{\|\hat{\underset{\sim}{x}}_C\| = a} V(\hat{\underset{\sim}{x}}_C, t) \tag{A3,A4}$$

This permits us to state the following:

Theorem (Ref. 20): The system described by Eqs. (44) is FTS with respect to $(\gamma, \beta, t_0, t_f, \| \ \|)$, $\gamma \le \beta$, if there exists a function $V(\hat{\underset{\sim}{x}}_C, t)$, as well as a function $\rho(t)$ integrable on T, such that:

a. $\dot{V}(\hat{\underset{\sim}{x}}_C, t) < \rho(t)$ for every $\hat{\underset{\sim}{x}}_C \epsilon B(\beta)$ and $t \epsilon (t_0, t_f)$

b. $\int_{t_0}^{t_f} \rho(\tau) d\tau \le V_m^\beta(t) - V_M^{B(\gamma)}(t_0)$, $t \epsilon (t_0, t_f)$

We choose

$$V(\hat{\underset{\sim}{x}}_C, t) = \hat{\underset{\sim}{x}}_C^T P \hat{\underset{\sim}{x}}_C \tag{A5}$$

Then, taking the time derivative and using Eqs. (44), (46) and (47), we obtain

$$\dot{V}(\hat{\underset{\sim}{x}}_C, t) = -\hat{\underset{\sim}{x}}_C^T (Q_1 + PGS^{-1}G^T P) \hat{\underset{\sim}{x}}_C \tag{A6}$$

But,

$$Q_1 + PGS^{-1}G^T P > 0,\quad t \epsilon T \tag{A7}$$

Hence, Condition a of the theorem is satisfied by choosing $\rho(t) \equiv 0$, so that Condition b reduces to

$$V_M^{B(\gamma)}(t_0) \le V_m^\beta(t),\quad t \epsilon T \tag{A8}$$

Considering Eq. (A5), Eq. (A8) yields

$$\lambda_M[P(t_0)]\gamma^2 = \max_{\|\hat{x}_C\| = \gamma} \hat{x}_C^T(t_0)P(t_0)\hat{x}_C(t_0) \le \min_{\|\hat{x}_C\| = \beta} \hat{x}_C^T P(t)\hat{x}_C = \lambda_m[P(t)]\beta^2 \quad \text{(A9)}$$

where λ_M and λ_m are the maximum and minimum eigenvalues of the argument matrix. We note here that all the eigenvalues of P are real and positive in T, because P is symmetric and positive definite.

Introducing $\|\hat{x}_C\| = \gamma$ in Eq. (A9), we conclude that

$$\|\hat{x}_C(t)\| < \beta \quad \text{(A11)}$$

for

$$\beta \le \|\hat{x}_C(t_0)\| \sqrt{\frac{\lambda_M[P(t_0)]}{\lambda_m[P(t)]}} \quad \text{(A10)}$$

Recalling Eq. (43a), letting $t_0 = 0$ and introducing the notation

$$\lambda_m^*[P] = \min_t \lambda_m[P(t)], \quad t \in T \quad \text{(A12)}$$

we obtain

$$\|x_C(t)\| < \|x_C(0)\| \sqrt{\frac{\lambda_M[P(0,\alpha)]}{\lambda_m^*[P(\alpha)]}}\, e^{-\alpha t}, \quad t \in T \quad \text{(A13)}$$

Inequality (A13) enables us to state that:

i. $\|x_C(t)\|$ approaches zero not slower than $e^{-\alpha t}$ and

ii. In most cases, the ratio of the eigenvalues in (A13) does not increase exponentially with α. Hence, increasing α causes $x_C(t)$ to decay to zero within the time interval t_f.

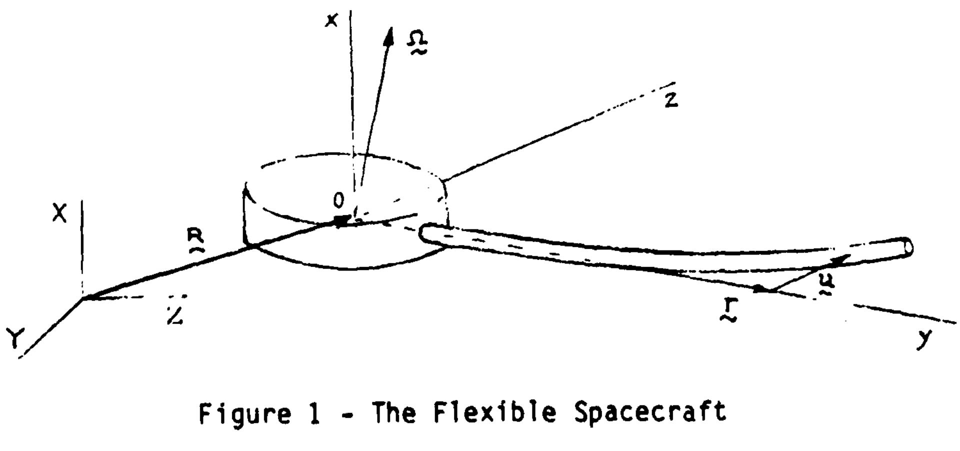

Figure 1 - The Flexible Spacecraft

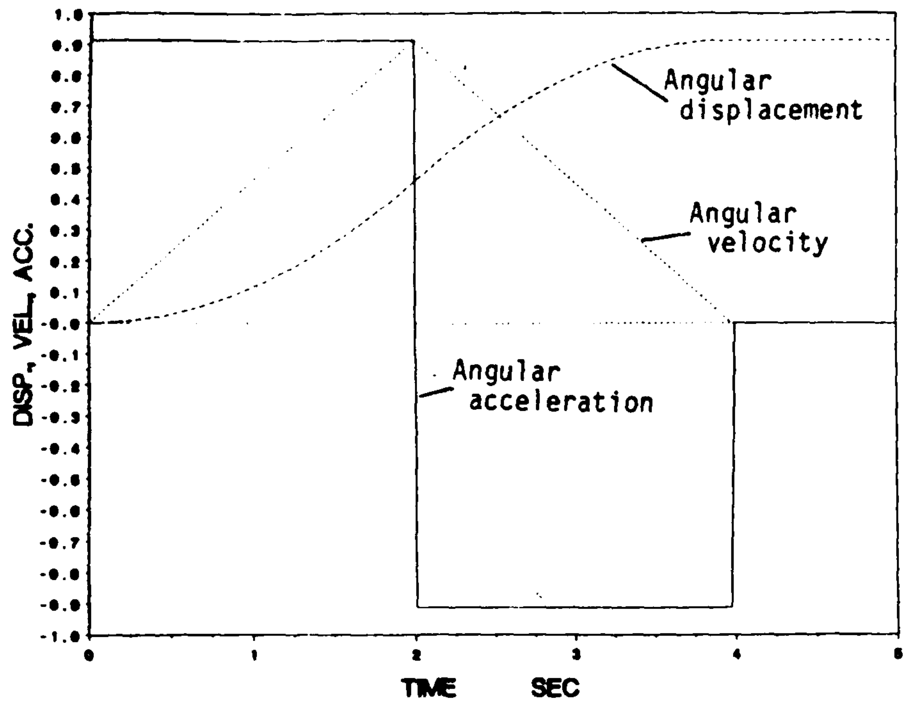

Figure 2 - Rigid-Body Maneuver (20°)

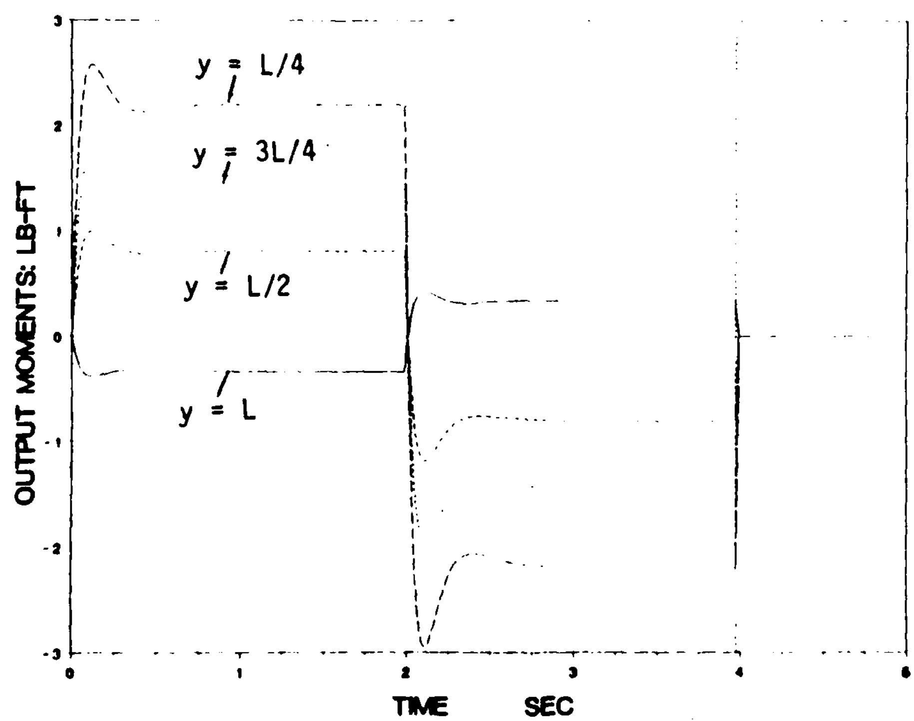

Figure 3 - Output Moments From Actuators on the Beam (20°)

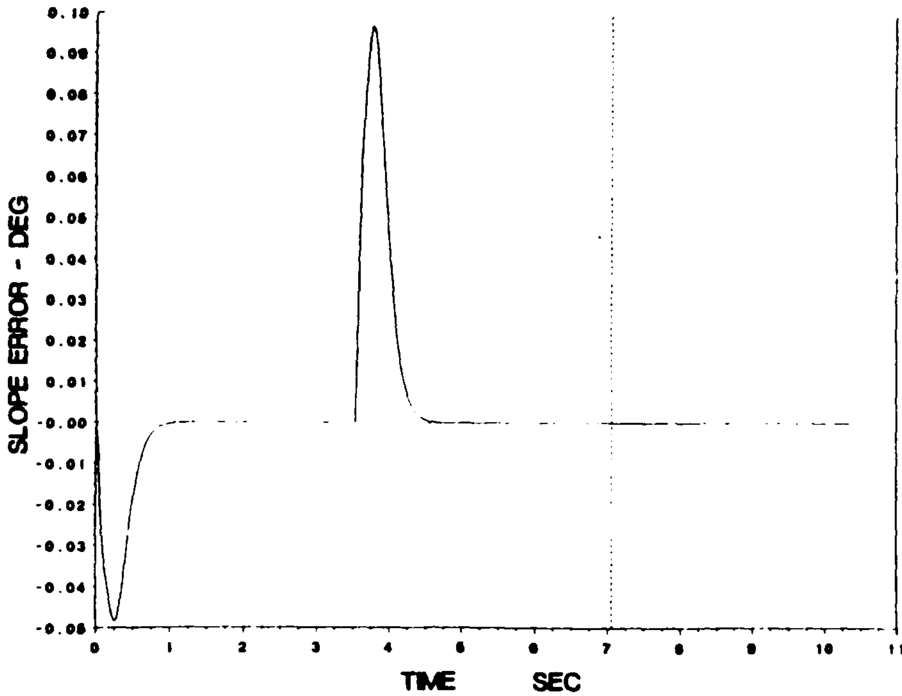

Figure 4 - Slope Error at the Tip of the Beam (180°)
Four Admissible Functions in the Model

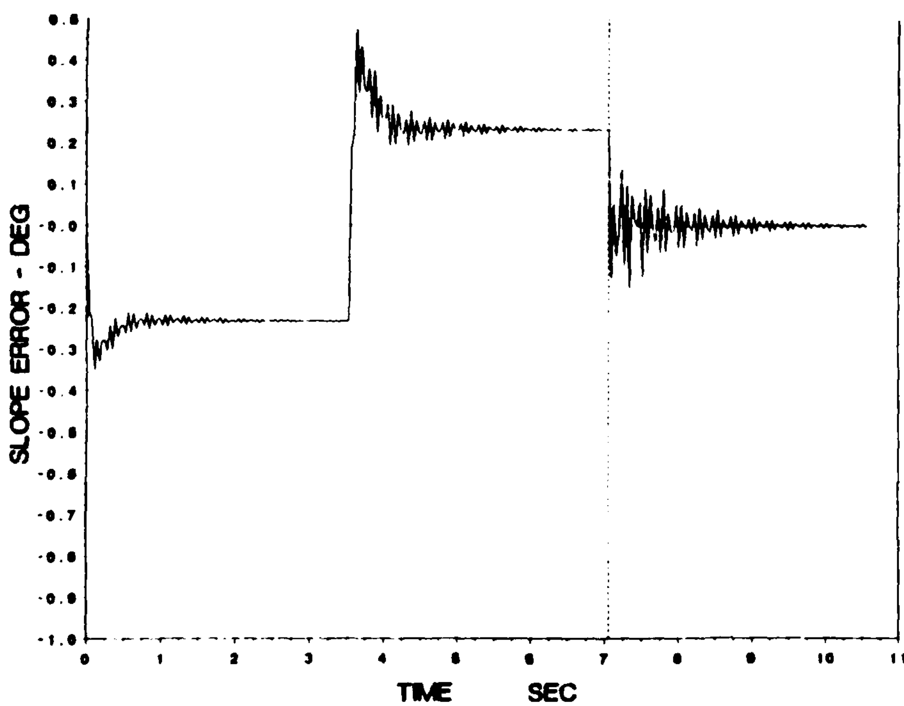

Figure 5 - Slope Error at the Tip of the Beam (180°)
Seven Admissible Functions in the Model

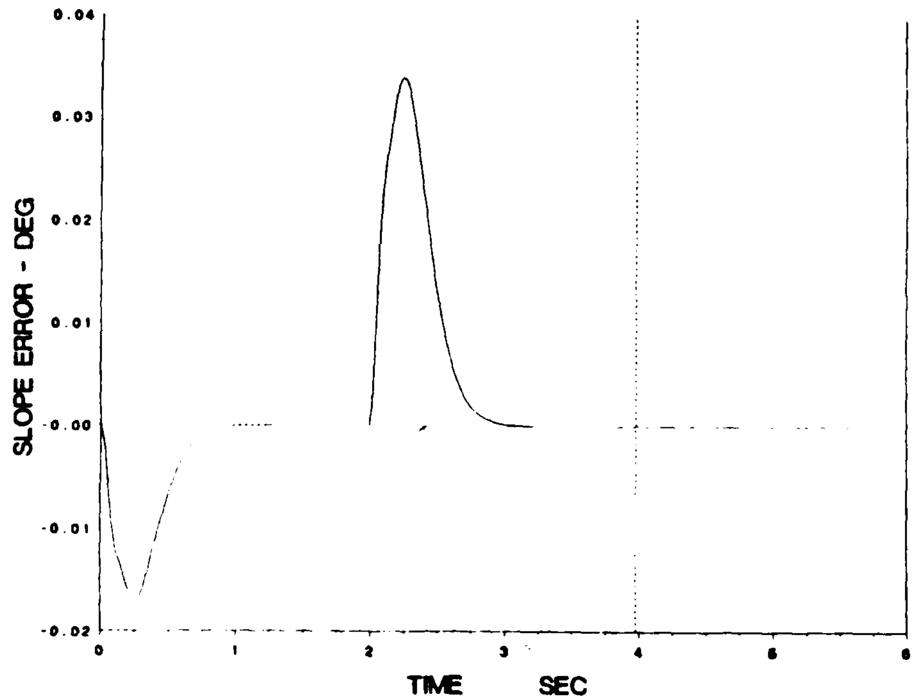

Figure 6 - Slope Error at the Tip of the Beam (20°)
Four Admissible Functions in the Model

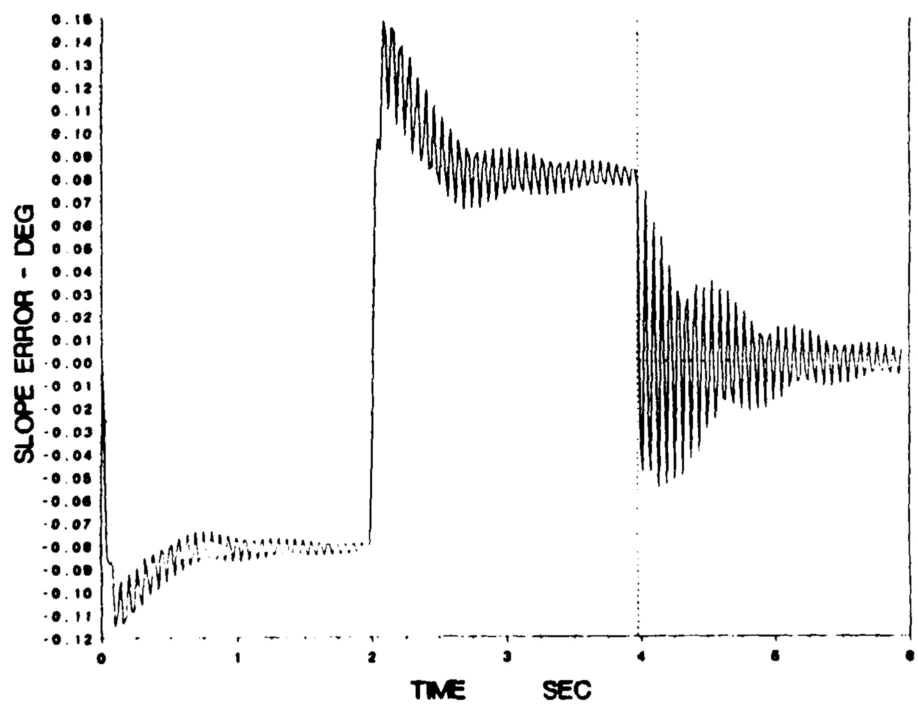

Figure 7 - Slope Error at the Tip of the Beam (20°)
Seven Admissible Functions in the Model

<u>CONTROL OF DISTRIBUTED STRUCTURES WITH SMALL NONPROPORTIONAL DAMPING*</u>

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Abstract

Undamped distributed structures represent self-adjoint systems, admitting real eigenvalues and real orthogonal eigenfunctions. Control of self-adjoint systems can be carried out conveniently by modal control. Distributed structures with proportional damping possess the same eigenfunctions as the corresponding undamped structures, so that the same modal approach can be used in this case as well. Nonproportional damping tends to destroy the self-adjointness of the system, so that modal control is not as convenient as for undamped structures. If damping is relatively small, however, it is possible to base the control design on the self-adjoint system and still obtain satisfactory control performance.

Introduction

Structures are distributed-parameter systems whose motion can be described by partial differential equations characterized by self-adjoint stiffness operators[1]. In the absence of damping, this implies that the structures themselves are self-adjoint, which implies further that the eigenfunctions are orthogonal. In general, damping tends to destroy the self-adjointness property. One notable exception is proportional damping, for which the eigenfunctions are the same as for the corresponding undamped structures, so that the orthogonality is preserved.

The self-adjointness property of distributed structures is very important in control design. Indeed, for undamped structures it is possible to take advantage of the eigenfunctions orthogonality and transform the partial differential equation into a set of independent second-order ordinary equations, known as modal equations. Then, controls can be designed for each modal equation independently, making it possible to target individual modes for control[2-5]. The same is true for systems with proportional damping. In the general case of feedback control of non-self-adjoint distributed structures, full decoupling of the distributed system may not be possible, so that modal control loses some of its appeal.

In many cases, the factors rendering the structures non-self-adjoint tend to be sufficiently small that they can be regarded as perturbations on the self-adjoint system. Then, the question arises as to the possibility of basing the controls on the self-adjoint system. This is the approach explored in this paper. A particular question to be answered is how large the factors rendering the system non-self-adjoint can be without invalidating the approach. This question is intimately related to that of robustness, i.e., how sensitive is the control design to changes in the system parameters, where these changes are the ones responsible for the non-self-adjointness of the system. One way of looking at the problem is by examining the closed-loop poles. In particular, the question is whether the factors rendering the structure non-self-adjoint can push the closed-loop poles far away from the closed-loop poles obtained from the control design based on the self-adjoint structure, thus impairing the control system performance.

Quite often, control of structures is carried out as if the structures were undamped. Inherent damping in the structure, however, causes the actual closed-loop poles to differ from the modeled closed-loop poles. It is shown in this paper that if the control system design takes advantage of the self-adjointness properties of structures, then the sensitivity of the closed-loop poles to the addition of small general viscous damping can be reduced to a simple algebraic expression. In fact, the closed-loop eigenvalue sensitivity turns out to be a function of the diagonal damping coefficients, so that the eigenvalues are insensitive to damping that might cause the system to become non-self-adjoint. A control system design which takes advantage of the self-adjoint property in structures is independent modal-space control (IMSC)[2-5].

This paper examines the circumstances under which controls based on a self-adjoint structure yield satisfactory performance when applied to the corresponding non-self-adjoint structure. To this end, the partial differential equation of motion for a distributed structure is first introduced. Then, a perturbation technique is used to compute the sensitivity of the closed-loop poles to the addition of small viscous damping in the system. The paper follows with an investigation of the sensitivity of the closed-loop poles when advantage is taken of the self-adjoint properties to design independent modal-space

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control for the structure. A numerical example is presented in which the control system performance is examined using varying amounts of damping. The numerical example indicates in a heuristic manner how large the nonproportional damping can be without degrading control system performance.

It is perhaps appropriate to mention another work using a perturbation approach to the control of structures[6]. The structure considered in Ref. 6 is undamped and the control is carried out by a low-authority controller, i.e., a controller providing sufficiently small damping that the modeled closed-loop poles could be computed from the open-loop eigensolution on the basis of a perturbation technique[1]. In contrast, the structure considered here is inherently damped, but the damping is sufficiently small that the controller can be designed by *ignoring the* damping. Then, the perturbation approach is used to calculate the shift in the undamped system closed-loop poles caused by the inherent damping.

Equations of Motion

The motion of a distributed structure can be derived using the extended Hamilton's principle[1] and is given by the partial differential equation

$$u(P,t) + \dot{u}(P,t) + m(P)\ddot{u}(P,t) = f(P,t), \quad P\varepsilon D \tag{1}$$

where $u(P,t)$ is the displacement at point P in the domain D of the structure, is a self-adjoint differential operator of order $2p$ expressing the system stiffness, is a viscous damping operator, $m(P)$ is a scalar function representing the mass distribution and $f(P,t)$ is the external force density. The displacement $u(P,t)$ is subject to the boundary conditions

$$B_i u(P,t) = 0, \quad i = 1,2,\ldots,p; \quad P\varepsilon S \tag{2}$$

where B_i are differential operators of maximum order $2p-1$ and S defines the boundary of D.

We consider *the eigenvalue problem* associated with the undamped self-adjoint structure given by

$$\phi_r(P) = \lambda_r m(P)\phi_r(P), \quad r = 1,2,\ldots; \quad P\varepsilon D \tag{3}$$

and the boundary conditions

$$B_i\phi_r(P) = 0, \quad i = 1,2,\ldots,p; \; r = 1,2,\ldots; \quad P\varepsilon S \tag{4}$$

The solution to Eqs. (3) and (4) consists of a denumerably infinite set of real nonnegative *eigenvalues* λ_r *and associated real eigenfunctions* $\phi_r(P)$. The eigenvalues are related to the natural frequencies of undamped vibration by $\lambda_r = \omega_r^2$ $(r = 1,2,\ldots)$. For convenience, we order the eigenvalues so that $\lambda_1 \leq \lambda_2 \leq \ldots$.

The eigenfunctions can be *normalized* so as to satisfy

$$\int_D m\phi_r(P)\phi_s(P)\,dD, \quad \int_D \phi_r(P) \quad \phi_s(P)\,dD = \omega_r^2\delta_{rs}, \qquad r,s = 1,2,\ldots \tag{5}$$

where δ_{rs} is the Kronecker delta function. Equations (5) represent the orthonormality conditions for *self-adjoint systems.*

We propose to express the solution of Eqs. (1) and (2) in terms of the infinite series

$$u(P,t) = \sum_{r=1}^{\infty} \phi_r(P)u_r(t) \tag{6}$$

where $u_r(t)$ are *time-dependent generalized* coordinates. Introducing Eq. (6) into (1), multiplying the result by ϕ_s, integrating over the domain D and considering Eqs. (5), we obtain

$$\ddot{u}_r(t) + \sum_{s=1}^{\infty} c_{rs}\dot{u}_s(t) + \omega_r^2 u_r(t) = f_r(t), \qquad r = 1,2,\ldots \tag{7}$$

where

$$c_{rs} = \int_D \phi_r \quad \phi_s\,dD, \quad r,s = 1,2,\ldots \tag{8}$$

are *viscous damping* coefficients and

$$f_r(t) = \int_D \phi_r(P)f(P,t)\,dD, \quad r = 1,2,\ldots \tag{9}$$

are generalized control forces.

Equations (7) represent an infinite set of *ordinary differential* equations expressing the motion of a distributed structure. In the absence of damping, $c_{rs} = 0$ $(r,s = 1,2,\ldots)$, the system is self-adjoint and Eqs. (7) represent a decoupled set. However, damping tends to prevent the decoupling and hence to destroy the self-adjointness property. In the case of non-self-adjoint systems it is necessary to first define and then solve an adjoint eigenvalue problem[1]. Only in special cases does the operator have ϕ_r $(r = 1,2,\ldots)$ as its eigenfunctions. In these cases, $c_{rs} = 0$ $(r\neq s)$. Note that when the operator admits ϕ_r $(r = 1,2,\ldots)$ as its eigenfunctions, the normal modes of the undamped structure are identical to the normal modes of the damped structure[1].

Equations (7) can be expressed in the vector form

$$\ddot{\underset{\sim}{u}}(t) + C\dot{\underset{\sim}{u}}(t) + \Lambda\underset{\sim}{u}(t) = \underset{\sim}{f}(t) \tag{10}$$

where

$$\underset{\sim}{u}(t) = [u_1(t)\; u_2(t)\; \ldots] \tag{11}$$

is an infinite-dimensional vector of generalized coordinates,

$$\underset{\sim}{f}(t) = [f_1(t)\; f_2(t)\; \ldots] \tag{12}$$

is an infinite-dimensional vector of generalized forces,

$$\Lambda = \text{diag}\,[\omega_1^2\ \omega_2^2\ \ldots] \tag{13}$$

is the $\infty\times\infty$ matrix of natural frequencies of the undamped structure squared and $C = [c_{rs}]$ is the damping matrix.

In general, the damping matrix C is fully populated. In this case, to obtain a solution, it is necessary to cast Eq. (10) in the state-space. To this end, we introduce the 2∞-dimensional state vector $\underset{\sim}{x}(t) = [\underset{\sim}{u}^T(t)\ \ \dot{\underset{\sim}{u}}^T(t)]^T$. Then, Eq. (10) can be written in the state vector form

$$\dot{\underset{\sim}{x}} = A^*\underset{\sim}{x} + B^*\underset{\sim}{f} \tag{14}$$

where

$$A^* = \left[\begin{array}{c|c} 0 & I \\ \hline -\Lambda & -C \end{array}\right],\quad B^* = \left[\begin{array}{c} 0 \\ \hline I \end{array}\right] \tag{15}$$

are coefficient matrices, in which I is an $\infty\times\infty$ identity matrix.

Sensitivy Analysis

We consider the case in which damping is sufficiently small that the control system design can be based on the undamped model, where the latter is obtained by letting C = 0 in Eq. (10). In general, the feedback control recouples the undamped equations. Indeed, the control law is given by

$$\underset{\sim}{f}(t) = -G\underset{\sim}{u} - H\dot{\underset{\sim}{u}} \tag{16}$$

where in general G, H are fully populated $\infty\times\infty$-dimensional control gain matrices with entries g_{rs}, h_{rs} $(r,s = 1,2,\ldots)$, respectively. The net effect of the feedback control is to recouple the undamped equations of motion, Eq. (10), through the nonzero off-diagonal terms in the gain matrices G and H. Substituting Eq. (16) into Eq. (10) with C = 0, we obtain the closed-loop state equations of motion in the vector form

$$\dot{\underset{\sim}{x}} = A_0\underset{\sim}{x} \tag{17}$$

where

$$A_0 = \left[\begin{array}{c|c} 0 & I \\ \hline -\Lambda - G & -H \end{array}\right] \tag{18}$$

The eigenvalue problems associated with Eq. (17) are

$$A_0\underset{\sim}{u}_{0r} = \lambda_{0r}\underset{\sim}{u}_{0r},\quad A_0^T\underset{\sim}{v}_{0s} = \lambda_{0s}\underset{\sim}{v}_{0s},\quad r,s = 1,2,\ldots \tag{19a,b}$$

where λ_{0r} $(r = 1,2,\ldots)$ are the closed-loop poles of the undamped system. We note that, because A_0 is not symmetric, we must consider both the right and left eigenvectors of A_0, denoted by $\underset{\sim}{u}_{0r}$ and $\underset{\sim}{v}_{0s}$, respectively. The two sets of eigenvectors $\underset{\sim}{u}_{0r}$, $\underset{\sim}{v}_{0s}$ are biorthogonal and can be normalized so as to satisfy the biorthonormality conditions[1]

$$\underset{\sim}{v}_{0s}^T\underset{\sim}{u}_{0r} = \delta_{rs},\quad \underset{\sim}{v}_{0s}^T A_0\underset{\sim}{u}_{0r} = \lambda_{0r}\delta_{rs},\quad r,s = 1,2,\ldots \tag{20a,b}$$

In general, the eigenvalues λ_{0r} $(r = 1,2,\ldots)$ and eigenvectors $\underset{\sim}{u}_{0r}$, $\underset{\sim}{v}_{0s}$ $(r,s = 1,2,\ldots)$ are complex quantities.

By virtue of the small-damping assumption, the open-loop actual distributed structure is nearly self-adjoint, and differs from the open-loop undamped structure by a small perturbation. Of course, the perturbation arises from the fact that the actual distributed structure contains small viscous damping. The actual closed-loop equations of motion are given by

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x} \tag{21}$$

where

$$A = \left[\begin{array}{c|c} 0 & I \\ \hline -\Lambda - G & -C - H \end{array}\right] \tag{22}$$

The eigenvalue problems associated with Eq. (21) have the same form as Eqs. (19), except that the subscript 0 is not present. Moreover, the two sets of eigenvectors $\underset{\sim}{u}_r$, $\underset{\sim}{v}_r$ can be normalized so as to satisfy the biothonormality conditions given by Eqs. (20) with the subscript 0 omitted.

When damping in the actual distributed structure is small, i.e., when $c_{rs} = O(\varepsilon)$ $(r,s = 1,2,\ldots)$, the matrix A can be expressed in the perturbed form

$$A = A_0 + A_1 \tag{23}$$

where A_0 is given by Eq. (18) and A_1 is a first-order perturbation matrix given by

$$A_1 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & -C \end{array}\right] \tag{24}$$

Because A is obtained from A_0 by a small perturbation, we assume that the eigensolutions can be written in the form

$$\lambda_r = \lambda_{0r} + \lambda_{1r},\quad r = 1,2,\ldots \tag{25a}$$

$$\underset{\sim}{u}_r = \underset{\sim}{u}_{0r} + \underset{\sim}{u}_{1r},\quad r = 1,2,\ldots \tag{25b}$$

$$\underset{\sim}{v}_r = \underset{\sim}{v}_{0r} + \underset{\sim}{v}_{1r},\quad r = 1,2,\ldots \tag{25c}$$

It can be shown that the eigenvalue and eigenvector perturbations have the expressions[1]

$$\lambda_{1r} = \underset{\sim}{v}_{0r}^T A_1\underset{\sim}{u}_{0r},\quad r = 1,2,\ldots \tag{26a}$$

$$\underset{\sim}{u}_{1r} = \sum_{k=1}^{\infty} \frac{\underset{\sim}{v}_{0k}^T A_1 \underset{\sim}{u}_{0r}}{\lambda_{0r} - \lambda_{0k}} \underset{\sim}{u}_{0k}(1 - \delta_{kr}) \tag{26b}$$

$$\underset{\sim}{v}_{1r} = \sum_{k=1}^{\infty} \frac{\underset{\sim}{u}_{0k}^T A_1 \underset{\sim}{v}_{0r}}{\lambda_{0r} - \lambda_{0k}} \underset{\sim}{v}_{0k}(1 - \delta_{kr}) \tag{26c}$$

Equations (26a) can be regarded as providing the sensitivity of the closed-loop poles to small viscous damping in the distributed structure. Indeed, dividing both sides of Eqs. (26a) by c_{ij}, we obtain the first-order eigenvalue sensitivity coefficient $\partial\lambda_r/\partial c_{ij}$, which is a measure of the change in the closed-loop eigenvalue λ_r due to the damping coefficient c_{ij}. In general, the closed-loop eigenvectors $\underset{\sim}{u}_{0r}$ and $\underset{\sim}{v}_{0r}$ (r = 1, 2,...) are fully populated so that the perturbations λ_{1r}, $\underset{\sim}{u}_{1r}$, $\underset{\sim}{v}_{1r}$ (r = 1,2,...) are sensitive to all the damping terms c_{rs} (r,s = 1,2,...).

Sensitivity Using Natural Control

Natural control is characterized by the preservation of the natural coordinates in the closed-loop system, which can be traced to the fact that the open-loop eigenfunctions are identical to the closed-loop eigenfunctions. It can be shown that natural control is the globally optimal solution to the control problem for self-adjoint distributed parameter systems[3]. Natural control is obtained using the independent modal space control (IMSC) method[2-5]. The IMSC control gain matrices G and H are diagonal with entries equal to g_r (r = 1,2,...) and h_r (r = 1,2,...), respectively. Hence, natural control can be regarded as providing proportional viscous damping and proportional stiffness augmentation.

Recognizing that the right eigenvectors have the form $\underset{\sim}{u}_{0r} = [\underset{\sim}{g}_{0r}^T \mid \lambda_{0r}\underset{\sim}{g}_{0r}^T]^T$, where $\underset{\sim}{g}_{0r}$ (r = 1, 2,...) are ∞-dimensional configuration eigenvectors, Eq. (19a) can be written in the explicit form

$$\left[\begin{array}{c|c} 0 & I \\ \hline -\Lambda - G & -H \end{array}\right]\left[\begin{array}{c} \underset{\sim}{g}_{0r} \\ \hline \lambda_{0r}\underset{\sim}{g}_{0r} \end{array}\right] = \lambda_{0r}\left[\begin{array}{c} \underset{\sim}{g}_{0r} \\ \hline \lambda_{0r}\underset{\sim}{g}_{0r} \end{array}\right] \tag{27}$$

It can be shown that when G and H are diagonal matrices, $\underset{\sim}{g}_{0r} = \underset{\sim}{e}_r$, where $\underset{\sim}{e}_r$ is a standard unit vector (with all its components equal to zero with the exception of the rth component which is equal to one). Then, from Eqs. (19b) and (20a), we obtain

$$\underset{\sim}{u}_{0r} = \left[\begin{array}{c} \underset{\sim}{e}_r \\ \hline \lambda_{0r}\underset{\sim}{e}_r \end{array}\right], \quad \underset{\sim}{v}_{0r} = \frac{1}{\alpha_r}\left[\begin{array}{c} (\omega_r^2 + g_r)\underset{\sim}{e}_r \\ \hline -\lambda_{0r}\underset{\sim}{e}_r \end{array}\right], \quad r = 1,2,... \tag{28a,b}$$

Hence, in the case of IMSC, characterized by diagonal control gain matrices G and H in Eq. (16), each of the closed-loop eigenvectors $\underset{\sim}{u}_{0r}$, $\underset{\sim}{v}_{0r}$ (r = 1,2,...) associated with the undamped structure contains only two nonzero terms. The closed-loop eigenvalues λ_{0r} and the normalization constants α_r (r = 1,2,...) have the expressions

$$\lambda_{0r} = -\frac{1}{2}\{h_r + i[4(\omega_r^2 + g_r) - h_r^2]^{1/2}\} \tag{29a}$$

$$\alpha_r = \omega_r^2 + g_r - \lambda_{0r}^2 \tag{29b}$$

Equations (28) and (29) specify only one half of the eigenvalues and left and right eigenvectors. The complex conjugates comprise the other half.

Inserting Eqs. (28) into (26a), we can write

$$\lambda_{1r} = \frac{1}{\alpha_r}\left[\begin{array}{c} (\omega_r^2 + g_r)\underset{\sim}{e}_r \\ \hline -\lambda_{0r}\underset{\sim}{e}_r \end{array}\right]^T \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & -C \end{array}\right]\left[\begin{array}{c} \underset{\sim}{e}_r \\ \hline \lambda_{0r}\underset{\sim}{e}_r \end{array}\right] = \frac{\lambda_{0r}^2}{\alpha_r} c_{rr}, \quad r = 1,2,... \tag{30}$$

where the real part of the quantity λ_{0r}^2/α_r (r = 1,2,...) is always negative. The perturbation in the right eigenvectors is obtained from Eq. (26b) and is given by

$$\underset{\sim}{u}_{1r} = -\sum_{k=1}^{\infty} \frac{1 - \delta_{kr}}{\lambda_{0r} - \lambda_{0k}} \frac{\lambda_{0r}\lambda_{0k}}{\alpha_k} c_{kr}\underset{\sim}{u}_{0k}, \quad r = 1,2,... \tag{31}$$

Examining Eq. (30), for a given closed-loop system, it is apparent that the perturbations in the closed-loop eigenvalues vary only as a function of the diagonal terms in the matrix C. Hence, damping perturbations in which $c_{rr} > 0$ (r = 1,2,...) cannot shift the actual structure closed-loop poles to the right in the complex plane relative to the corresponding closed-loop poles of the undamped structure. Note that this conclusion was reached on the basis of natural control. The off-diagonal terms in the damping matrix are the terms inducing the actual distributed structure to vibrate in a linear combination of complex modes, which are different from the undamped normal modes. However, when the damping terms are small, as is the case here, the closed-loop poles are sensitive only to changes in the damping matrix which do not induce non-self-adjoint properties. Moreover, when the damping matrix C is diagonally-dominant, the perturbations in the eigenvectors given by Eqs. (31) are of second-order in magnitude, and the closed-loop system tends to vibrate essentially as a linear combination of its normal modes only.

Numerical Example

To illustrate the sensitivity of the closed-loop poles and control system performance to small viscous damping, we consider the control of

the bending vibration of a simply-supported beam. Assuming that the beam has uniform mass and stiffness properties, Eq. (1) becomes[7]

$$EI\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2}{\partial x^2}\left[cI(x)\ \frac{\partial^3 u}{\partial x^2 \partial t}\right] + M\frac{\partial^2 u}{\partial t^2} = f(x,t), \quad 0 < x < L \qquad (32)$$

where we note that the damping operator in Eq. (32), given by $= \frac{\partial^2}{\partial x^2}\left[cI(x)\ \frac{\partial^2}{\partial x^2}\right]$, provides strain rate damping, which is a form of viscous damping. The displacement u is subject to the boundary conditions

$$u(0,t) = \frac{\partial^2 u(0,t)}{\partial x^2} = u(L,t) = \frac{\partial^2 u(L,t)}{\partial x^2} = 0 \qquad (33)$$

For simplicity, we choose $M = 1$, $EI = 1$ and $L = 5$.

It can be shown that the solution of the associated eigenvalue problem, Eq. (3), for the undamped case, i.e., for $cI(x) = 0$, consists of the eigenfunctions and natural frequencies[8],

$$\phi_r(x) = \left(\frac{2}{L}\right)^{1/2}\sin\left(\frac{r\pi x}{L}\right), \quad r = 1,2,\ldots \qquad (34a)$$

$$\omega_r = \left(\frac{r\pi}{L}\right)^2, \quad r = 1,2,\ldots \qquad (34b)$$

respectively. For the damped case, we choose the strain rate damping distribution to be given by

$$cI(x) = \alpha + \beta x(L - x), \quad 0 < x < L \qquad (35)$$

where α and β are constants. Note that when $\beta = 0$, the strain rate damping is uniform, which results in the case of proportional damping, as in this case the damping distribution is proportional to the stiffness distribution. However, the presence of a nonzero β destroys the self-adjointness property, producing nonzero off-diagonal entries in the damping matrix C. From Eq. (8), we obtain upon integrations by parts

$$c_{rs} = \int_0^L \phi_r(x)\frac{d^2}{dx^2}\left\{[\alpha + \beta x(L - x)]\frac{d^2\phi_s}{dx^2}\right\} dx$$

$$= \int_0^L [\alpha + \beta x(L - x)]\frac{d^2\phi_r(x)}{dx^2}\frac{d^2\phi_s(x)}{dx^2} dx \qquad (36)$$

With the eigenfunctions given by Eq. (34a), the damping coefficients in Eq. (36) can be computed in closed-form and are given by

$$c_{rs} = \begin{cases} \alpha\left(\frac{r\pi}{L}\right)^4 + \frac{2\beta(r\pi)^4}{L^2}\left(\frac{1}{12} + \frac{1}{4r^2\pi^2}\right), \\ \qquad r = 1,2,\ldots,;r = s \\ -\frac{\beta s^2 r^2 \pi^2}{L^2}\left[\frac{(-1)^{r-s} - 1}{(r-s)^2} - \frac{(-1)^{r+s} - 1}{(r+s)^2}\right], \\ \qquad r,s = 1,2,\ldots,;r \neq s \end{cases} \qquad (37)$$

Note that with $\beta = 0$, we have proportional damping, in which case we obtain $c_{rs} = \alpha\omega_r^2\delta_{rs} = \alpha\lambda_r\delta_{rs}$.

We consider controlling a subset of the modes using IMSC, as controlling the entire infinity of modes would require a distributed actuator[2]. More specifically, the first three modes are to be controlled using 3 discrete actuators at locations $x_i = iL/4$ $(i = 1,2,3)$ together with modal filters[2], which are used for the extraction of the modal coordinates. To determine the control gain matrices G and H in Eq. (16), the following performance index is minimized

$$J = \sum_{r=1}^{3}\int_0^\infty [\dot{u}_r^2(t) + \lambda_r u_r^2(t) + R_r f_r^2(t)]\, dt \qquad (38)$$

Note that the first two terms in the integrals in Eq. (38) represent modal kinetic and potential energies, respectively, and the third term represents the modal control effort, all for the first three modes. The minimization yields the modal gains[2]

$$g_r = -\omega_r^2 + \omega_r(\omega_r^2 + 1/R_r)^{1/2}, \quad r = 1,2,3 \qquad (39a)$$

$$h_r = [-2\omega_r^2 + 1/R_r + 2\omega_r(\omega_r^2 + 1/R_r)^{1/2}]^{1/2}, \quad r = 1,2,3 \qquad (39b)$$

The control law given by Eq. (16) with the control gains provided by Eqs. (39) represents linear optimal IMSC.

To show the effects of damping, we investigate the 3 cases, $\alpha = \beta = 0$, $\alpha = \beta = 0.0001$ and $\alpha = \beta = 0.001$, denoted by I, II and III, respectively. Table 1 displays the damping matrix with entries given by Eq. (37) with $\alpha = \beta = 1$, where the actual distributed structure is simulated by truncating the infinite-dimensional system by retaining 8 terms in series (6). Table 2 provides the closed-loop poles for the 3 cases with $R_r = 1.0$ $(r = 1,2,3)$. Examining Figure 1, we note that damping shifts the closed-loop poles to the left in the complex plane as the level of damping increases. Morever, the shift in the closed-loop poles for the controlled modes, i.e., the lowest 3 modes, can be computed using Eqs. (29) and (30), where the natural frequencies ω_r $(r = 1,2,\ldots)$ are given by Eq. (34b) and the control gains g_r, h_r $(r = 1,2,3)$ are computed using Eqs. (39) with $R_r = 1.0$ $(r = 1,2,3)$.

To examine the control system performance, we considered Cases I, II and III, as above, except that we increased the control effort by decreasing R_r in Eq. (38). To this end, we used $R_r = 0.01$ $(r = 1,2,3)$, which has the effect of increasing the gains, as can be seen from Eqs. (39). Figures 2 and 3 plot the performance index $J(t)$ given by Eq. (38) for the values $R_r = 1.0$ and $R_r = 0.01$ $(r = 1,2,3)$, respectively, for the above three cases. It is evident from Figure 2 that for $R_r = 1.0$ the effect of damping is to

enhance the control system performance, as the performance index J(t) has the values 893.1, 883.7 and 813.1 at t = 5 s corresponding to the Cases I, II and III, respectively. Figure 3, however, displays the opposite effect, with the performance index J(t) for $R_r = 0.01$ having the values 250.3, 250.4 and 252.3 at t = 5 s corresponding to the Cases I, II and III, respectively. Hence, damping may enhance or degrade control system performance depending on the value of the gains. Indeed, as the gains increase, larger amounts of energy are transferred back and forth between the controlled and uncontrolled modes, which are coupled due to the presence of damping. This effect is known as control spillover[9]. However, we also conclude that the damping does not alter control system performance significantly.

Conclusions

Quite often, control of structures is carried out as if the structures were undamped. Undamped structures are self-adjoint. Inherent structural damping tends to destroy the self-adjointness properties and can degrade control system performance. However, designing controls on the basis of an undamped structures has many advantages. One of these advantages is that is permits the use of IMSC. It turns out that in this case the perturbation in the modeled closed-loop poles due to small damping does not cause a dramatic shift in the closed-loop poles nor does it change the control system performance very much. The sensitivity study presented here indicates that damping shifts the actual closed-loop poles to the left of the modeled closed-loop poles in the complex plane. Moreover, the presence of small damping in the actual distributed structure does not affect the control system performance significantly in the case of IMSC.

References

1. Meirovitch, L., Computational Methods in Structural Dynamics, Sijthoff & Noordhoff, The Netherlands, 1980.

2. Meirovitch, L. and Baruh, H., "Control of Self-Adjoint Distributed-Parameter Systems," Journal of Guidance and Control, Vol. 5, No. 1, 1982, pp. 60-66.

3. Meirovitch, L. and Silverberg, L. M., "Globally Optimal Control of Self-Adjoint Distributed Systems," Optimal Control Applications and Methods, Vol. 4, 1983, pp. 365-386.

4. Meirovitch, L., Baruh, H. and Öz, H., "A Comparison of Control Tehcniques for Large Flexible Systems," Journal of Guidance, Control, and Dynamics, Vol. 6, No. 4, 1983, pp. 302-310.

5. Meirovitch, L., Baruh, H., Montgomery, R. C. and Williams, J. P., "Nonlinear Natural Control of an Experimental Beam," Journal of Guidance, Control, and Dynamics, Vol. 7, No. 4, 1984, pp. 437-442.

6. Aubrun, J. N. "Theory of the Control of Structures by Low-Authority Controllers," Journal of Guidance and Control, Vol. 3, No. 5, 1980, pp. 444-451.

7. Clough, R. W. and Penzien, J., Dynamics of Structures, McGraw-Hill Book Company, New York, New York, 1975.

8. Meirovitch, L., Analytical Methods in Vibrations, The Macmillan Company, New York, New York, 1967.

9. Balas, M. J., "Active Control of Flexible Systems," Journal of Optimization Theory and Applications, Vol. 25, No. 3, 1978, pp. 415-436.

Table 1. Damping Matrix for $\alpha = \beta = 1$.

2.27	7.02	0.00	2.25	0.00	1.39	0.00	1.02
7.02	30.44	68.22	.00	17.90	0.00	10.70	0.00
0.00	68.22	148.57	278.44	0.00	63.17	0.00	36.08
2.25	0.00	278.44	463.41	779.82	0.00	159.16	0.00
0.00	17.90	0.00	779.82	1124.42	1761.85	0.00	332.23
1.39	0.00	63.17	0.00	1761.85	2323.79	3461.39	0.00
0.00	10.70	0.00	159.16	0.00	3461.39	4296.37	6162.70
1.02	0.00	36.08	0.00	332.23	0.00	6162.70	7319.76

Table 2. Closed-Loop Poles for $R_r = 1.0$ ($r = 1,2,3$).

Case I	Case II	Case III
$\alpha = \beta = 0$	$\alpha = \beta = 0.0001$	$\alpha = \beta = 0.001$
-0.6199 ± 1.0057	-0.6200 ± 1.0056	-0.6211 ± 1.0050
-0.6921 ± 1.8436	-0.6936 ± 1.8430	-0.7076 ± 1.8382
-0.7037 ± 3.6792	-0.7112 ± 3.6778	-0.7815 ± 3.6676
± 6.3165	-0.0231 ± 6.3168	-0.2248 ± 6.3379
± 9.8696	-0.0562 ± 9.8704	-0.5643 ± 9.9485
± 14.2122	-0.1162 ± 14.2146	-1.1630 ± 14.4489
± 19.3444	-0.2148 ± 19.3504	-1.8443 ± 20.4288
± 25.2662	-0.3660 ± 25.2453	-3.9635 ± 22.6735

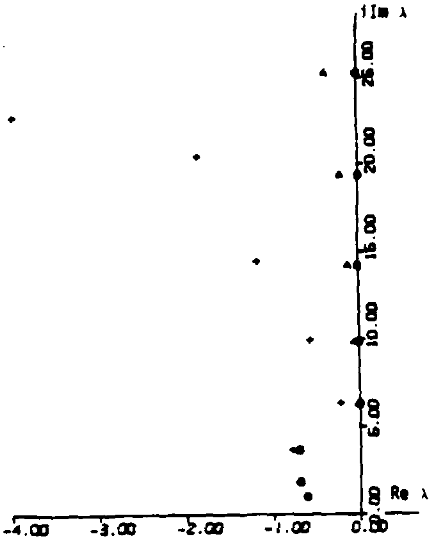

Figure 1. Closed-Loop Poles for the Cases I(o), II(Δ) and III(+)

Case I: $\alpha = \beta = 0$
Case II: $\alpha = \beta = 0.0001$
Case III: $\alpha = \beta = 0.001$

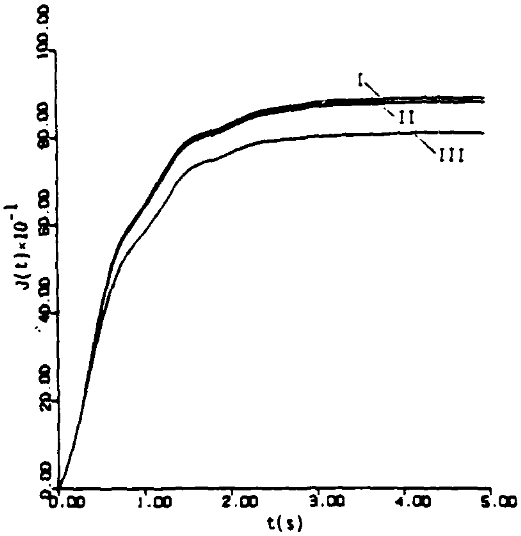

Figure 2. Performance Index for Cases I, II and III for $R_r = 1$ ($r = 1,2,3$)

Case I: $\alpha = \beta = 0$
Case II: $\alpha = \beta = 0.0001$
Case III: $\alpha = \beta = 0.001$

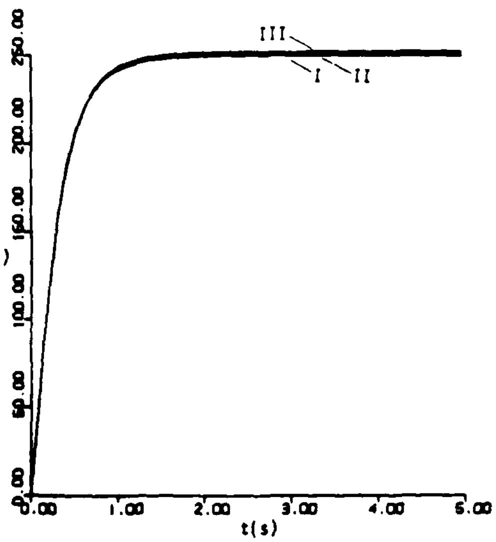

Figure 3. Performance Index for Cases I, II and III for $R_r = 0.01$ ($r = 1,2,3$)

Case I: $\alpha = \beta = 0$
Case II: $\alpha = \beta = 0.0001$
Case III: $\alpha = \beta = 0.001$